Solutions to the Fall 2003 prelim

1A. Show that the differential equation

$$f''(z) = zf(z), \qquad f(0) = 1, \qquad f'(0) = 1$$

has an unique entire solution in the complex plane.

Solution. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be the Taylor series of f at 0. Then the equation gives

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0$$

$$k(k-1)a_k = a_{k-3}.$$

Hence for $k \geq 1$ we obtain

$$a_{3k} = \prod_{j=1}^{k} \frac{1}{3j(3j-1)}$$

$$a_{3k+1} = \prod_{j=1}^{k} \frac{1}{3j(3j+1)}$$

$$a_{3k+2} = 0.$$

We need to show that the convergence radius of the series for f is infinite. Indeed we have

$$\lim_{k\to\infty} \frac{a_{3k+3}}{a_{3k}} = 0$$

which shows that the series

$$\sum_{k=0}^{\infty} a_{3k} z^{3k}$$

has an infinite radius of convergence. Similarly we argue for the "$3k+1$" series.

2A. List eight groups of order 36 and prove that they are not isomorphic.

Solution. Let C_n be a cyclic group of order n, let $D_{2\cdot n}$ be a dihedral group of order $2n$, let S_n be the symmetric group on n letters, and let A_n be its alternating subgroup. Consider the following eight groups of order 36:

$$\begin{array}{cccc} C_2^2 \times C_3^2 & C_2^2 \times C_9 & C_4 \times C_3^2 & C_4 \times C_9 \\ C_6 \times S_3 & S_3 \times S_3 & C_2 \times D_{2\cdot 9} & C_3 \times A_4. \end{array}$$

The first four are abelian and pairwise nonisomorphic because each pair has either distinct 2-Sylow subgroups or distinct 3-Sylow subgroups. They are not isomorphic to the last four because the latter are nonabelian.

Of the last four, only $C_2 \times D_{2\cdot 9}$ has a cyclic 3-Sylow subgroup, only $C_3 \times A_4$ has a normal 2-Sylow subgroup, and only $S_3 \times S_3$ has a trivial center. Thus the last four also are pairwise nonisomorphic.

(Remark: in fact, there are 14 groups of order 36.)

3A. Let A be a 2×2 matrix with complex entries. Prove that the series $I + A + A^2 + \dots$ converges if and only if every eigenvalue of A has absolute value less than 1.

Solution. Conjugating A changes neither the convergence nor the eigenvalues, so we may assume that A is in Jordan canonical form, i.e., $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

In the first case, $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ and $\sum A^n$ converges if and only if the eigenvalues a and b have absolute value less than 1, because the entries of the sum are geometric series.

In the second case, write $A = aI + N$, so $N^2 = 0$, and $A^n = a^n I + na^{n-1}N$. If $I + A + A^2 + \dots$ converges, then the diagonal entries a^n of the terms A^n must converge to 0, so $|a| < 1$. Conversely if $|a| < 1$, then $\sum a^n$ and $\sum na^{n-1}$ converge by the Ratio Test, so $\sum A^n$ converges.

4A. Give an example, with proof, of a nonconstant irreducible polynomial $f(x)$ over $\mathbb{Q}$ with the property that $f(x)$ does not factor into linear factors over the field $K = \mathbb{Q}[x]/(f(x))$.

Solution. The simplest example is $f(x) = x^3 - 2$. Let $\sqrt[3]{2}$ denote the real cube root of 2. Then $\mathbb{Q}(\sqrt[3]{2})$ is an algebraic extension of $\mathbb{Q}$ generated by a root of $x^3 - 2$, hence isomorphic to $K = \mathbb{Q}[x]/(x^3 - 2)$. Since $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$, and $x^3 - 2$ has only one real root, $x^3 - 2$ does not factor completely over K. The same proof works with $f(x) = x^3 - a$ for any rational a that is not a cube of a rational number. Other examples are also possible, of course.

5A. Let C denote the space of continuous functions on $[0, 1]$. Define

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

(a) Show that d is a metric on C.
(b) Show that (C, d) is not a complete metric space.

Solution. The function $a \mapsto a/(1 + a) = 1 - 1/(1 + a)$ is increasing on $[0, \infty)$. Hence, for $a = |f - g|$, $b = |g - h|$, $c = |f - h|$, we have $c \le a + b$ and

$$\frac{c}{1+c} \le \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b}.$$

This implies the triangle inequality.

Define

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ 1/x, & 1/n \le x \le 1. \end{cases}$$

The f_n form a Cauchy sequence, since

$$\begin{aligned} d(f_m, f_n) &= \int_0^{\max\{1/m, 1/n\}} \frac{|f_m(x) - f_n(x)|}{1 + |f_m(x) - f_n(x)|}\, dx \\ &\le \int_0^{\max\{1/m, 1/n\}} 1\, dx \\ &= \max\{1/m, 1/n\}. \end{aligned}$$

Suppose that (C, d) is a complete metric space. Then the f_n would converge to some $f \in C$. If $f(a) \neq 1/a$ for some $a \in (0,1]$, then by continuity there exists $\epsilon > 0$ such that $|1/x - f(x)| \geq \epsilon$ for $x \in (a-\epsilon, a]$. Then

$$d(f_n, f) \geq \int_{a-\epsilon}^{a} \frac{\epsilon}{1+\epsilon}\, dx$$

for sufficiently large n. But the right hand side is a positive constant independent of n, so then f_n could not converge to f. Thus $f(a) = 1/a$ for all $a \in (0,1]$. This contradicts the fact that f is continuous on $[0,1]$.

6A. Let $A(m,n)$ be the $m \times n$ matrix with entries

$$a_{ij} = j^i \quad (0 \leq i \leq m-1,\ 0 \leq j \leq n-1),$$

where $0^0 = 1$ by definition. Regarding the entries of $A(m,n)$ as representing congruence classes (mod p), determine the rank of $A(m,n)$ over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for all $m, n \geq 1$ and all primes p.

Solution. The upper-left $k \times k$ square minor $A(k,k)$ of $A(m,n)$ is the Vandermonde matrix, with determinant $\prod_{0 \leq i < j < k}(j-i)$. If $k \leq p$, this determinant is non-zero (mod p), which shows that $\operatorname{rk} A(m,n) \geq \min(m,n,p)$. Conversely, $A(m,n)$ has at most p distinct columns (mod p), so $\operatorname{rk} A(m,n) \leq p$. Since $\operatorname{rk} A(n,n) \leq \min(m,n)$, we have $\operatorname{rk} A(m,n) = \min(m,n,p)$.

7A. Let $D = \{z \in \mathbb{C} : |z| \leq 1\} - \{1, -1\}$. Find an explicit continuous function $f : D \to \mathbb{R}$ satisfying all the following conditions:

- f is harmonic on the interior of D (the open unit disk),
- $f(z) = 1$ when $|z| = 1$ and $\operatorname{Im}(z) > 0$, and
- $f(z) = -1$ when $|z| = 1$ and $\operatorname{Im}(z) < 0$.

Solution. The linear fractional transformation $w = (1+z)/(1-z)$ maps $|z| < 1$ to the half-plane $\operatorname{Re}(w) > 0$, with the upper and lower boundary semicircles mapping to the half-lines $i\mathbb{R}_{>0}$ and $i\mathbb{R}_{<0}$, respectively. A branch of $\log w$ defined on $\mathbb{C} - \mathbb{R}_{\leq 0}$ has

$$\operatorname{Im}(\log w) = \begin{cases} \pi/2, & w \in i\mathbb{R}_{>0} \\ -\pi/2, & w \in i\mathbb{R}_{<0}, \end{cases}$$

so $f(z) = \frac{2}{\pi}\operatorname{Im}(\log((1+z)/(1-z)))$ is a solution.

8A. Let p be a prime, and let G be the group $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. How many automorphisms does G have?

Solution. An automorphism of G is determined by where it sends the generators $(1,0)$ and $(0,1)$. We claim that for $(a,b), (c,d) \in G$, there exists an automorphism mapping $(1,0)$ to (a,b) and $(0,1)$ to (c,d) if and only if

$$a \notin p\mathbb{Z}/p^2\mathbb{Z}, \quad c \in p\mathbb{Z}/p^2\mathbb{Z}, \quad \text{and} \quad d \neq 0 \in \mathbb{Z}/p\mathbb{Z}.$$

If α is an automorphism mapping $(1,0)$ to (a,b) and $(0,1)$ to (c,d), then (a,b) must not be killed by p, so $a \notin p\mathbb{Z}/p^2\mathbb{Z}$ and (c,d) must be killed by p, so $c \in p\mathbb{Z}/p^2\mathbb{Z}$. Moreover (c,d) should not be a multiple of $p(a,b) = (pa,0)$, so $d \neq 0$.

Conversely, given (a,b) and (c,d) satisfying the conditions, there exists a homomorphism $\alpha : G \to G$ mapping $(1,0)$ to (a,b) and $(0,1)$ to (c,d), since (a,b) is killed by p^2 and (c,d) is killed by p. The condition on a implies that (a,b) has order p^2. If (c,d) were a multiple of (a,b), then since $c \in p\mathbb{Z}/p^2\mathbb{Z}$, the element (c,d) would be a multiple of $p(a,b) = (pa,0)$, which is impossible, since $d \neq 0 \in \mathbb{Z}/p\mathbb{Z}$. Thus $\#\alpha(G) > p^2$. so by Lagrange's Theorem $\#\alpha(G) = p^3$. Thus α is surjective, but G is finite, so α is also injective, so α is an automorphism.

It remains to count (a,b,c,d) satisfying the conditions. There are $p^2 - p$ possibilities for a, p possibilities for b, p possibilities for c, and $p-1$ possibilities for d, and these may be chosen independently, so in total there are $(p^2-p)p^2(p-1) = p^5 - 2p^4 + p^3$ automorphisms of G.

9A. Let $f : [0,1] \to [0,1]$ be an increasing (not strictly increasing) function such that

$$f\left(\sum_{j=1}^{\infty} a_j 3^{-j}\right) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$$

whenever the a_j are 0 or 2. Prove that there is a constant C_0 such that

$$|f(x) - f(y)| \le C_0 |x-y|^{(\log 2)/(\log 3)}$$

for all $x, y \in [0,1]$.

Solution. Let $x = 0.a_1a_2\ldots$ in base 3. If $a_j = 1$ for some j, choose the smallest such j, and define

$$\begin{aligned} x_- &= 0.a_1a_2\ldots a_{j-1}022222\ldots \\ x_+ &= 0.a_1a_2\ldots a_{j-1}200000\ldots. \end{aligned}$$

(These are the nearest numbers in C on either side of x, where C is the Cantor set consisting of numbers in $[0,1]$ representable by base-3 expansions with only 0's and 2's.) Then $f(x_-) = f(x_+)$, so f is constant on $[x_-, x_+]$.

Thus it suffices to prove the inequality with $x = \sum a_j 3^{-j} \ge y = \sum b_j 3^{-j}$ with $a_j, b_j \in \{0,2\}$. Let $\hat{j}$ be the smallest j with $a_j \neq b_j$. Then $|x-y| \ge 3^{-\hat{j}}$. On the other hand,

$$|f(x) - f(y)| = \left| \sum_{j \ge \hat{j}} \frac{a_j - b_j}{2} 2^{-j} \right| \le \sum_{j \ge \hat{j}} 2^{-j} = 2 \cdot 2^{-\hat{j}}.$$

Combining, we obtain

$$|f(x) - f(y)| \le 2 \cdot 2^{-\hat{j}} \le 2(3^{-\hat{j}})^{(\log 2)/(\log 3)} \le 2|x-y|^{(\log 2)/(\log 3)}.$$

1B. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^n+1}\,dx$, where $n \geq 4$ is an even integer.

Solution. Let $f(x)$ be the integrand. The answer is $2I$, where $I := \int_0^\infty f(x)\,dx$. For $R > 1$, let γ_R be the straight line path from 0 to R, followed by the arc Re^{it} for $t \in [0, 2\pi/n]$, followed by the straight line path from $Re^{2\pi i/n}$ back to 0.

Let $\zeta = e^{\pi i/n}$. The poles of $f(z)$ are at ζ^{2m+1} for $m \in \mathbb{Z}$, so the only pole inside γ_R is ζ. The numerator is nonzero at ζ, while the denominator has nonzero derivative at ζ, so ζ is a simple pole with residue

$$\frac{\zeta^2}{n\zeta^{n-1}} = \frac{1}{n}\zeta^{3-n}.$$

By the residue theorem,

$$\int_{\gamma_R} f(z)\,dz = \frac{2\pi i}{n}\zeta^{3-n} = -\frac{2\pi i}{n}\zeta^3.$$

On the other hand, the first straight part of the integral tends to I as $R \to \infty$, the curved part of the integral tends to 0 as $R \to \infty$ since the integrand is $O(1/R^{n-2}) \leq O(1/R^2)$ while the length of the arc is $O(R)$, and the last straight part of the integral tends to $-\zeta^6 I$ as $R \to \infty$, as the substitution $z = \zeta^2 t$ shows. Thus

$$I - \zeta^6 I = -\frac{2\pi i}{n}\zeta^3.$$

Now

$$\sin(3\pi/n) = \frac{\zeta^3 - \zeta^{-3}}{2i} = \frac{\zeta^6 - 1}{2i\zeta^3},$$

so

$$\begin{aligned} 2I &= \frac{4\pi i}{n} \cdot \frac{\zeta^3}{\zeta^6 - 1} \\ &= \frac{4\pi i}{n} \cdot \frac{1}{2i\sin(3\pi/n)} \\ &= \frac{2\pi}{n\sin(3\pi/n)}. \end{aligned}$$

2B. Let $u_{m,n}$ be an array of numbers for $1 \leq m \leq N$ and $1 \leq n \leq N$. Suppose that $u_{m,n} = 0$ when m is 1 or N, or when n is 1 or N. Suppose also that

$$u_{m,n} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}\right)$$

whenever $1 < m < N$ and $1 < n < N$. Show that all the $u_{m,n}$ are zero.

Solution. If not, then by changing signs, we may assume that $M := \max u_{m,n}$ is positive. Let

$$R = \{(m,n) : u_{m,n} = M\} \subseteq \{2, 3, \ldots, N-1\} \times \{2, 3, \ldots, N-1\}.$$

Choose $(m,n) \in R$ with m minimal. Since $(m-1,n) \notin R$,

$$\frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}\right) < \frac{1}{4}(M + M + M + M) = M = u_{m,n}.$$

This contradicts the given relation.

3B. Let A and B be $n \times n$ complex unitary matrices. Prove that $|\det(A+B)| \le 2^n$.

Solution. Let $C = A^{-1}B$, which also is unitary. Then

$$A + B = A(I + C)$$

Since A is unitary, its eigenvalues have absolute value 1. Multiplying them together shows that $|\det A| = 1$. If $\zeta_1, \ldots, \zeta_n$ are the eigenvalues of C with multiplicity, so $|\zeta_i| = 1$, then the eigenvalues of $I + C$ are $1 + \zeta_1, \ldots, 1 + \zeta_n$, so

$$|\det(I + C)| = |1 + \zeta_1| \ldots |1 + \zeta_n| \le 2 \cdot 2 \ldots 2 = 2^n$$

Thus

$$|\det(A + B)| = |\det(A)||\det(I + C)| \le 2^n.$$

4B. Let L be a line in $\mathbb{C}$, and let f be an entire function such that $f(\mathbb{C}) \cap L = \emptyset$. Prove that f is constant. (Do not use the theorem of Picard that the image of a nonconstant entire function omits at most one complex number.)

Solution. Replacing f by $f + c$ for some $c \in \mathbb{C}$, we may assume that $0 \in L$. Replacing f by αf for some $\alpha \in \mathbb{C}^*$, we may assume that L is the imaginary axis. Since $f(\mathbb{C})$ is connected, it is contained in either the right half plane or the left half plane. Replace f by $-f$ if necessary, to assume that $f(\mathbb{C})$ is contained in the left half plane. Then $g(z) = e^{f(z)}$ is entire and bounded, hence it is a constant c by Liouville's theorem. Then $f(\mathbb{C})$ is contained in the set of solutions to $e^z = c$, which is discrete, but $f(\mathbb{C})$ is connected, so $f(\mathbb{C})$ must be a point. Thus f is constant.

5B. Let n be a positive integer. Let $\phi(n)$ be the Euler phi function, so $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^*$. Prove that if $\gcd(n, \phi(n)) > 1$, then there exists a noncyclic group of order n.

Solution. Let p be a prime dividing both n and $\phi(n)$. The formula for $\phi(n)$ shows that either $p^2|n$ or there is a different prime $q|n$ such that $p|(q-1)$.

If $p^2|n$, then $C_p \times C_p \times C_{n/p^2}$ is a noncyclic group of order n (where C_m denotes a cyclic group of order m).

In the other case, let G be the subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a^p = 1$. Since $\mathbb{F}_q^*$ is cyclic of order $q - 1$, there are p solutions to $a^p = 1$ in $\mathbb{F}_q$. Thus $\#G = pq$. If $a^p = 1$ and $a \ne 1$, then

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ne \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

so G is not abelian. Then $G \times C_{n/pq}$ has order n and is not cyclic (since it is not abelian).

6B. Let $f(z)$ be a meromorphic function on the complex plane. Suppose that for every polynomial $p(z) \in \mathbb{C}[z]$ and every closed contour Γ avoiding the poles of f, we have

$$\int_\Gamma p(z)^2 f(z)\, dz = 0.$$

Prove that $f(z)$ is entire.

Solution. Comparing the condition with $p(z)$ replaced by $p(z)+1$ and subtracting, we find that

$$\int_\Gamma (2p(z)+1)f(z)\,dz = 0.$$

Every polynomial can be written as $2p(z)+1$, so we have that

$$\int_\Gamma p(z)f(z)\,dz = 0$$

for every polynomial $p(z)$.

Suppose that $f(z)$ has a pole of order n at $a \in C$. Then $(z-a)^{n-1}f(z)$ has a nonzero residue at a, so

$$\int_\Gamma (z-a)^{n-1}f(z)\,dz \neq 0$$

for a sufficiently small loop Γ around a. Thus $f(z)$ cannot have any poles. Hence $f(z)$ is entire.

7B. (a) Let G be a finite group and let X be the set of pairs of commuting elements of G:

$$X = \{(g,h) \in G \times G : gh = hg\}.$$

Prove that $|X| = c|G|$ where c is the number of conjugacy classes in G.

(b) Compute the number of pairs of commuting permutations on five letters.

Solution. (a) Let C_g denote the conjugacy class of g and Z_g the centralizer of g. By the orbit-stabilizer theorem, we have $|Z_g| \cdot |C_g| = |G|$ for every g. Hence $\sum_{g\in C} |Z_g| = |G|$ for every conjugacy class C, and $|X| = \sum_{g\in G} |Z_g| = c|G|$.

(b) Take $G = S_5$, with $|G| = 5! = 120$. The number of conjugacy classes c is the number of partitions of 5, namely 7. So there are $7 \cdot 120 = 840$ pairs of commuting permutations.

8B. The set of 5×5 complex matrices A satisfying $A^3 = A^2$ is a union of conjugacy classes. How many conjugacy classes?

Solution. A matrix A is a solution to $x^3 = x^2$ (or equivalently, $x^2(x-1) = 0$) if and only if all its Jordan blocks are. In particular, each Jordan block must have eigenvalues 0 and 1, and the possible Jordan blocks are

$$(0), \quad (1), \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The conjugacy type of a matrix is determined by the multiplicities of the Jordan blocks. Let a, b, c be the multiplicities of the blocks above, respectively. Then the answer is the number of nonnegative integer solutions to

$$a + b + 2c = 5.$$

For fixed $c \in \{0,1,2\}$, there are $6-2c$ solutions to $a+b = 5-2c$. Thus the answer is

$$(6 - 2\cdot 0) + (6 - 2\cdot 1) + (6 - 2\cdot 2) = 12.$$

9B. Let $\lambda, a \in \mathbb{R}$, with $a > 0$. Let $u(x, y)$ be an infinitely differentiable function defined on an open neighborhood of $x^2 + y^2 \leq 1$ such that

$$\begin{aligned} \Delta u + \lambda u &= 0 && \text{in } x^2 + y^2 < 1 \\ u_n &= -au && \text{on } x^2 + y^2 = 1. \end{aligned}$$

Here Δ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and u_n denotes the directional derivative of u in the direction of the outward unit normal (pointing away from the origin). Prove that if u is not identically zero in $x^2 + y^2 < 1$, then $\lambda > 0$.

Solution. Let D be the closed unit disk. Then

$$\int_D u(\Delta u + \lambda u) = \int_D 0 = 0.$$

If we substitute

$$u\,\Delta u = \underline{\nabla} \cdot (u \underline{\nabla} u) - |\underline{\nabla} u|^2,$$

this becomes

$$\int_D \underline{\nabla} \cdot (u \underline{\nabla} u) - \int_D |\underline{\nabla} u|^2 + \int_D \lambda u^2 = 0.$$

Applying the Divergence Theorem (in the form

$$\int_D \underline{\nabla} \cdot \underline{f} = \int_{\partial D} \underline{f} \cdot \underline{n}$$

where $\underline{n}$ is the outward unit normal) to the first term, we get

$$\int_{\partial D} u u_n - \int_D |\underline{\nabla} u|^2 + \int_D \lambda u^2 = 0.$$

Since $u_n = -au$ on ∂D, we get

$$-a \int_{\partial D} u^2 - \int_D |\underline{\nabla} u|^2 + \lambda \int_D u^2 = 0.$$

Since u is not identically zero on D, we have $\int_D u^2 > 0$. If u were constant on D, the equation $u_n = -au$ on ∂D would force $u = 0$. Thus $\underline{\nabla} u$ is not identically zero on D, so $\int_D |\underline{\nabla} u|^2 > 0$. Finally, $a \int_{\partial D} u^2 \geq 0$. Thus solving for λ shows that $\lambda > 0$.