FALL 2006 PRELIMINARY EXAMINATION SOLUTIONS

1A. Compute

$$\lim_{x\to 0} \frac{d^4}{dx^4} \frac{x}{\sin x}.$$

Solution: By Taylor's formula,

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5).$$

Therefore

$$\begin{aligned}\frac{x}{\sin x} &= \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)} \\ &= 1 + (\frac{x^2}{6} - \frac{x^4}{120} + o(x^4)) + (\frac{x^2}{6} + o(x^2))^2 + o(x^4) \\ &= 1 + \frac{x^2}{6} + \left[\frac{1}{36} - \frac{1}{120}\right] x^4 + o(x^4).\end{aligned}$$

Thus,

$$\lim_{x\to 0} \frac{d^4}{dx^4} \frac{x}{\sin x} = 4! \left[\frac{1}{36} - \frac{1}{120}\right] = \frac{2}{3} - \frac{1}{5} = \frac{7}{15}.$$

2A. Let

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

Compute

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Solution: The matrix A has eigenvalues 2 and 1 with eigenvectors $(2, 1)$ and $(1, 1)$ respectively. Therefore

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Observe that

$$e^{CBC^{-1}} = \sum_{n=0}^{\infty} \frac{(CBC^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{CB^nC^{-1}}{n!} = Ce^BC^{-1}.$$

Therefore

$$e^A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2e^2 & e \\ e^2 & e \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2e^2 - e & -2e^2 + 2e \\ e^2 - e & -e^2 + 2e \end{pmatrix}.$$

3A. Let U be a connected open subset of $\mathbb{C}$ containing -2 and 0. Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function whose Taylor expansion at 0 is $\sum_{n\geq 0} \binom{2n}{n} z^n$. Prove that $f(-2) \in$

$\{1/3, -1/3\}$. (Note: The original version of this problem had an error: $\{3, -3\}$ instead of $\{1/3, -1/3\}$.)

Solution: We claim that $f(z)^2 = (1-4z)^{-1}$. Since a holomorphic function on a connected open set is determined by its values on any nonempty open subset, it suffices to prove $f(z)^2 = (1-4z)^{-1}$ in a neighborhood of 0.

One way to do this is to expand $(1-4z)^{-1/2}$ using the binomial theorem, and check that it agrees with $\sum_{n\geq 0}\binom{2n}{n}z^n$. But this assumes that we guessed the formula $(1-4z)^{-1/2}$.

A more motivated solution is to find a differential equation satisfied by $f(z)$ (in a neighborhood of 0). Rewrite the series as

$$f = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!}(2z)^n,$$

where $(2n-1)!!$ denotes the product of all odd positive integers up to $2n-1$. The series satisfies the 1st order differential equation:

$$z\frac{d}{dz}f = 2z(2z\frac{d}{dz}+1)f.$$

It can be rewritten as

$$\frac{df}{dz} = \frac{2f}{1-4z},$$

which is not hard to solve:

$$\int\frac{df}{f} = \int\frac{2dz}{1-4z}, \text{ or } \ln f = -\frac{1}{2}\ln(1-4f) + const,$$

i.e. $f = C(1-4z)^{-1/2}$. The value $C = 1$ is found from $f(0) = 1$.

Now $f(-2)^2 = (1-4(-2))^{-1} = 1/9$, so $f(-2) \in \{1/3, -1/3\}$.

4A. Let R be a finite commutative ring without zero-divisors and containing at least one element other than 0. (As usual, rings are associative with 1.) Prove that R is a field.

Solution: Let $a \in R$, $a \neq 0$ and let $f : R \to R$ be $f(x) = ax$, $x \in R$. Then f is one-to-one since there are no zero-divisors in R. Then f is onto since R is finite. Thus there exists a unique $x_a \in R$ such that $ax_a = a$. Let us show that x_a plays the role of unity in R. Indeed, for every $b \in R$ there is a unique $x_b \in R$ such that $b = ax_b$. We have $bx_a = ax_bx_a = ax_ax_b = ax_b = b$. So $x_a = 1$. For each $0 \neq b \in R$ there is a unique b' with $bb' = 1$. Thus $b' = b^{-1}$.

5A. Let $C^0[0,1]$ be the vector space over $\mathbb{R}$ consisting of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that the linear operator $T\colon C^0[0,1] \to C^0[0,1]$ defined by

$$(Tf)(x) := \int_0^x f(y)\,dy$$

has no nonzero eigenvectors.

Solution: Suppose that $f \in C^0[0,1]$ and $\lambda \in \mathbb{R}$ satisfy $Tf = \lambda f$. By the fundamental theorem of calculus, Tf is differentiable, and its derivative is $(Tf)' = f$. Therefore $\lambda f' = f$. Solving this differential equation (e.g. by separation of variables), we find that if $\lambda = 0$ then

$f = 0$, while if $\lambda \neq 0$ then $f = Ce^{x/\lambda}$. But we observe that $(Tf)(0) = 0$, so in the case when $\lambda \neq 0$ we have $C = 0$. Either way, $f = 0$. Thus T has no nonzero eigenvector.

6A. Let p be prime. Prove that the polynomial $f(x) = x^p - x + 1$ is irreducible over the field $\mathbb{F}_p$ of p elements.

Solution: Let α be a zero of f in some field extension of $\mathbb{F}_p$. Because of the identity $(x+y)^p = x^p + y^p$ in characteristic p, we have $f(x+1) = f(x)$. By induction, $f(x+a) = f(x)$ for all $a \in \mathbb{F}_p$. In particular, $f(\alpha + a) = f(\alpha) = 0$. Thus the p elements $\alpha + a$ for $a \in \mathbb{F}_p$ are all the zeros of $f(x)$.

Suppose $f(x) = g(x)h(x)$ for some monic polynomials $g, h \in \mathbb{F}_p[x]$. Then $g(x) = \prod_{i \in I}(x - (\alpha + i))$ for some subset $I \subseteq \mathbb{F}_p$. The sum of the zeros of g is in $\mathbb{F}_p$, so

$$(\#I)\alpha + (\sum_{i \in I} i) \in \mathbb{F}_p.$$

Thus $(\#I)\alpha \in \mathbb{F}_p$. Since f is irreducible, $\alpha \notin \mathbb{F}_p$, so $\#I$ must be divisible by p. In other words, $\#I$ is 0 or p, so the factorization is trivial.

7A. Prove that for every $a \in \mathbb{C}$ and integer $n \geq 2$, the equation $1 + z + az^n = 0$ has at least one root in the disk $|z| \leq 2$.

Solution: 1) If $a = 0$, the problem is trivial.

2) Let $a \neq 0$, $b = \dfrac{1}{a}$. Consider

$$b + bz + z^n = 0. \tag{1}$$

Let $z_1, \ldots, z_n$ be the roots of (1).

a) If $|b| \leq 2^n$ then there is z_i such that $|z_i| \leq 2$, since otherwise we would have $|b| = |z_1 \ldots z_n| > 2^n$.

b) Let $|b| > 2^n$ and let $f(z) = b(1+z) + z^n$, $g(z) = b(1+z)$. Then $|f(z) - g(z)| = |z^n| = 2^n < |b| = |b|(|z| - 1) \leq |b(1+z)| = |g(z)|$ if $|z| = 2$. By Rouché's Theorem, the function f has as many roots inside the circle $|z| = 2$ as does the function $g(z)$. But $g(z)$ has one, namely $z = -1$. Hence f also has one inside $|z| = 2$.

8A. Let $\mathbf{Z}$ denote the ring of integers and consider the linear map $\mathbf{Z}^3 \to \mathbf{Z}^3$ defined by the 3×3-matrix

$$A = \begin{pmatrix} 6 & 9 & 12 \\ 6 & 9 & 12 \\ 12 & 18 & 24 \end{pmatrix}$$

Compute the structure of the three abelian groups kernel(A), image(A), and cokernel$(A) = \mathbf{Z}^3/\text{image}(A)$. In particular, in each case determine whether the group is free abelian. If yes, give a basis.

Solution: We perform elementary row and column operations to diagonalize the matrix A:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} -1 & -3 & -2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both transformation matrices have determinant one, so they are invertible over the integers. Hence image, kernel and cokernel can be computed from the transformed matrix. We find

$$\text{image}(A) \simeq \mathbf{Z}^1, \quad \text{kernel}(A) \simeq \mathbf{Z}^2, \quad \text{coker}(A) \simeq \mathbf{Z}^2 \oplus \mathbf{Z}/2\mathbf{Z}.$$

We see that the column vector $(3,3,6)^T$ is a basis for image(A). The last two columns of the right transformation matrix give the basis $\{(-3,2,0)^T, (-2,0,1)^T\}$ for kernel(A).

9A. Let k be a field such that the additive group of k is finitely generated. Prove that k is finite.

Solution: First suppose that k has characteristic 0. A subgroup of a finitely generated abelian group is also finitely generated, so if the additive group of k is finitely generated, then so is the additive group of $\mathbb{Q}$. But the additive group generated by a finite list of rational numbers $a_1/b_1, \ldots, a_n/b_n$ is contained in the integer multiples of $1/(b_1 \cdots b_n)$, so if p is a prime larger than $|b_1 \cdots b_n|$, then $1/p$ is not in this group. This contradiction shows that k cannot have characteristic 0.

Let p be the characteristic of k. Then k is a vector space over the field $\mathbb{F}_p$ of p elements. Now, to say that k is finitely generated as an additive group is the same as saying that it is finite-dimensional as an $\mathbb{F}_p$-vector space. If $d = \dim_{\mathbb{F}_p} k$, then $\#k = p^d$, so k is finite.

1B. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that $|f(z^2)| \le 2|f(z)|$ for all $z \in \mathbb{C}$. Show that f is constant.

Solution: By induction on n we have that $|f(z^{2^n})| \le 2^n|f(z)|$.
(proof: $n = 0$ says $|f(z^1)| \le 1|f(z)|$; if this is true for n then:
$|f(z^{2^{n+1}})| = |f((z^{2^n})^2)| \le 2|f(z^{2^n})| \le 2(2^n)|f(z)|$).

Let $M = \max\{|f(z)| \;:\; |z| = 2\}$. Let $R_n = 2^{2^n}$. If $|w| = R_n$ then $w = z^{2^n}$ for some z of length 2, and so $|f(w)| \le 2^n|f(z)| \le 2^n M$.

For each integer $m \ge 1$, by Cauchy's inequalities for the circle about 0 of radius R_n, $|f^{(m)}(0)| \le (2^n M)/(R_n)^m \le M(2^{n-2^n})$. But as $n \to \infty$, this converges to 0. So $f^{(m)}(0) = 0$ for all $m \ge 1$, and the power series of f is constant.

2B. Let $C^0[0,1]$ be the vector space over $\mathbb{R}$ consisting of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that the functions $1, x, x^2, \ldots$ are linearly independent in $C^0[0,1]$.

Solution: Suppose that a finite linear combination $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ is equal to zero in $C^0[0,1]$, where $c_0, \ldots, c_n \in \mathbb{R}$. This means that $p(x) = 0$ for all $x \in [0,1]$. We need to show that $c_0 = \cdots = c_n = 0$. Pick any $n+1$ distinct points $a_1, \ldots, a_{n+1} \in [0,1]$. Since $p(a_1) = 0$, we have $p(x) = (x - a_1)q(x)$ where q is a polynomial of degree $n-1$. Since $p(a_2) = 0$ and $a_2 - a_1 \neq 0$, we have $q(a_2) = 0$, so the polynomial q is divisible by $x - a_2$. Continuing, we find that the polynomial p is divisible by $(x - a_1) \cdots (x - a_{n+1})$, and since the latter polynomial has degree $n+1$, this is possible only if $p = 0$.

3B. Let $f \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous function. For $x \in \mathbb{R}$, define

$$g(x) := \max\{f(x,y) : y \in [0,1]\}.$$

Show that g is continuous.

Solution: Given $a \in \mathbb{R}$, let $A = [a-1, a+1]$. $K = A \times [0,1]$ is compact, so f restricted to K is uniformly continuous. Given $\epsilon > 0$, let $\delta > 0$ be such that for all $x, z \in A$, $|x - z| < \delta$ implies for all $y \in [0,1]$, $\quad |f(x,y) - f(z,y)| < \epsilon$.

So for $x, z \in A$, if $|x - z| < \delta$, then $g(x) < g(z) + \epsilon$
(proof: Let y be such that $f(x,y) = g(x)$; so $|f(x,y) - f(z,y)| < \epsilon$, and $g(x) = f(x,y) < f(z,y) + \epsilon \le g(z) + \epsilon$).

By symmetry, for $x, z \in A$, if $|x - z| < \delta$ then $|g(x) - g(z)| < \epsilon$. So g is continuous at a.

4B. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial. Suppose there is a field extension F of $\mathbb{Q}$ containing a root a of $f(x)$ such that F does not contain any cube root of a. Show that $f(x^3)$ is irreducible over $\mathbb{Q}$.

Solution: Let $n = \deg f$. Let b be a root of $f(x^3)$ in some field extension of $\mathbb{Q}$. So b^3 is a root of $f(x)$. Since $f(x)$ is irreducible over $\mathbb{Q}$, the fields $\mathbb{Q}(a)$ and $\mathbb{Q}(b^3)$ are isomorphic via an isomorphism that sends a to b^3. Thus $\mathbb{Q}(b^3)$ contains no root of the polynomial $x^3 - b^3$. Since this is a cubic polynomial, this implies that $x^3 - b^3$ is irreducible over $\mathbb{Q}(b^3)$. Thus $[\mathbb{Q}(b) : \mathbb{Q}(b^3)] = 3$. Since $f(x)$ is of degree n and irreducible over $\mathbb{Q}$, $[\mathbb{Q}(b^3) : \mathbb{Q}] = n$.

So $[\mathbb{Q}(b) : \mathbb{Q}] = [\mathbb{Q}(b) : \mathbb{Q}(b^3)][\mathbb{Q}(b^3) : \mathbb{Q}] = 3n =$ the degree of $f(x^3)$. Thus $f(x^3)$ is the irreducible polynomial of b over $\mathbb{Q}$.

5B. Let f and g be entire functions such that

$$\int_{|z|=1} \frac{f(z)}{(\sin z)^m}\, dz = \int_{|z|=1} \frac{g(z)}{(\sin z)^m}\, dz$$

for all positive integers m. Prove that $f = g$.

Solution: Suppose $f \neq g$. Let $h(z) = f(z) - g(z)$, so

$$\int_{|z|=1} \frac{h(z)}{(\sin z)^m}\, dz = 0.$$

Since h is not identically zero, we may take $m = 1 + \operatorname{ord}_{z=0} h(z)$. Then $h(z)/(\sin z)^m$ has a simple pole at $z = 0$ and is holomorphic elsewhere in $|z| \le 1$, so the residue theorem gives

$$\int_{|z|=1} \frac{h(z)}{(\sin z)^m}\, dz \neq 0,$$

a contradiction.

6B. Let G be a nonabelian group of order 21. Find the largest positive integer n with the property that whenever G acts on a set S of size n, some element of S is fixed by every element of G.

Solution: Finite G-sets are finite unions of transitive G-sets, and each transitive G-set is of the form G/H for some subgroup H (namely, H is the stabilizer of a point in the G-set). Hence an integer n does *not* have the property if and only if there is a sequence of proper subgroups of G whose indices sum to n. The possibilities for the index of a proper subgroup of G are 3, 7, and 21 (consider Sylow subgroups, and the trivial group). Thus we seek the largest n that is not a sum of integers each of which equals 3, 7, or 21. The set of such sums consists of numbers of the form $3k$, numbers of the form $3k+1$ that are at least 7, and numbers of the form $3k+2$ that are at least 14, so the largest n that is not such a sum is 11.

7B. Let X and Y be metric spaces, and let $f_1, f_2, \ldots$ be continuous functions from X to Y. Suppose that the sequence $\{f_n\}$ converges uniformly to a function f. Show that f is continuous.

Solution: Let $\epsilon > 0$ and $x \in X$ be given; we must find $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. Since the sequence $\{f_n\}$ converges uniformly to f, there exists n such that for all $x \in X$ we have $d(f_n(x), f(x)) < \epsilon/3$. Since f_n is continuous, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f_n(x), f_n(x')) < \epsilon/3$. In particular, $d(x, x') < \delta$ implies that

$$\begin{aligned} d(f(x), f(x')) &\le d(f(x), f_n(x)) + d(f_n(x), f_n(x')) + d(f_n(x'), f(x')) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

8B. Let A be an $n \times n$ Hermitian matrix and B an $n \times n$ positive definite (complex) matrix. Prove that there is an invertible complex $n \times n$ matrix S such that $S^H AS$ is diagonal and $S^H BS = I$. (Here S^H denotes the conjugate transpose of the matrix S.)

Solution: Since B is positive definite there is a unitary V such that $B = VDV^H$ where D is diagonal with positive diagonal. Let $Q = V(\sqrt{D})^{-1}$. Then $Q^H BQ = (\sqrt{D})^{-1} V^H BV (\sqrt{D})^{-1} = I$. Then $Q^H AQ$ is Hermitian hence there is a unitary U such that $U^H (Q^H AQ) U = \Lambda$ is diagonal. Set $S = QU$. We have $S^H BS = U^H Q^H BQU = I$ and $S^H AS = U^H Q^H AQU = \Lambda$.

9B. Let $z_0, z_1, \ldots$ be a sequence of complex numbers such that $z_{n+1} = 1 + 1/z_n$ for all $n \ge 0$. Prove that the sequence is convergent.

Solution: Let $f(z) = \frac{z+1}{z}$. Then the equation $f(z) = z$ has two solutions

$$\alpha = \frac{1+\sqrt{5}}{2}, \ \beta = \frac{1-\sqrt{5}}{2}.$$

Let

$$w = \frac{z-\alpha}{z-\beta}, \ z = \frac{\beta w - \alpha}{w-1}.$$

Then

$$\frac{f(z)-\alpha}{f(z)-\beta} = \frac{z+1-\alpha z}{z+1-\beta z}.$$

Use $\alpha + \beta = 1$ and $\alpha\beta = -1$,

$$\frac{z+1-\alpha z}{z+1-\beta z} = \frac{\beta z + 1}{\alpha z + 1} = \frac{\beta}{\alpha}\frac{z-\alpha}{z-\beta} = \frac{\beta}{\alpha} w.$$

Therefore if $z_{n+1} = f(z_n)$, then $w_{n+1} = \gamma w_n$, where $\gamma = \frac{\beta}{\alpha}$. Since $|\gamma| < 1$,

$$\lim_{n\to\infty} w_n = 0,$$

that implies

$$\lim_{n\to\infty} z_n = \alpha$$

for any z_0, except $z_0 = \beta$. If $z_0 = \beta$, obviously the limit is β.