FALL 2007 PRELIMINARY EXAMINATION

1A. Let $\mathbb{Z}[i]$ be the set of complex numbers of the form $a + bi$ where a and b range over all integers. List all subrings of $\mathbb{Z}[i]$. (Your list should contain each subring exactly once.)

2A. Let $f(z)$ and $g(z)$ be entire functions such that $f'(z) = g(z)$, $g'(z) = -f(z)$, and $f(2z) = 2f(z)g(z)$ for all $z \in \mathbb{C}$. Find all possibilities for $f(z)$.

3A. Let A be an $n \times n$ Hermitian matrix, and let $x \in \mathbb{C}^n$ be a vector such that $A^2x = 0$. Prove that $Ax = 0$.

4A. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers. Suppose that $0 \leq a_{n+1} \leq a_n + b_n$ for all $n \geq 1$, and that $\sum_{n=1}^{\infty} b_n$ converges. Prove that $\lim_{n\to\infty} a_n$ exists and is finite.

5A. Suppose that G is a finite group such that for each subgroup H of G there exists a homomorphism $\phi\colon G \to H$ such that $\phi(h) = h$ for all $h \in H$. Show that G is a product of groups of prime order.

6A. Let $f(z) = z^4 + \dfrac{z^3}{4} - \dfrac{1}{4}$. How many zeros does f have in $\{z \in \mathbb{C} : \dfrac{1}{2} < |z| < 1\}$?

7A. Let $P \in \mathbb{R}^{n\times n}$ be a matrix satisfying $P^3 = P$. Let r be the rank of P and assume $r > 0$. Show that there exist matrices $U, V \in \mathbb{R}^{n\times r}$ satisfying $V^T U = I_r$ such that

$$P = USV^T,$$

where I_r is the $r \times r$ identity matrix, and S is an $r \times r$ diagonal matrix with ± 1's on the diagonal.

8A. Suppose that $(b_n)_{n\geq 1}$ is a sequence of positive real numbers tending to infinity such that $b_n/n \to 0$. Must there exist a sequence $(a_n)_{n\geq 1}$ such that $(a_1 + \cdots + a_n)/n \to 0$ and $\limsup_{n\to\infty}(a_n/b_n) = \infty$?

9A. Let G be a non-abelian group of order 16 having a subgroup H isomorphic to $C_2 \times C_2 \times C_2$ (where C_2 denotes a cyclic group of order 2). Prove that the number of elements of G of exact order 2 is either 7 or 11.

1B. Let $f(z)$ be a polynomial with complex coefficients, and let a be a complex number. Prove that $\{a, f(a), f(f(a)), \ldots\}$ is not dense in $\mathbb{C}$.

2B. Let A be an $n \times n$ complex matrix. Suppose that m is a positive integer such that A^m is diagonalizable. Prove that A^{m+1} is diagonalizable.

3B. Let $\{u_1, u_2, \cdots, u_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, and let $\mathcal{A}$ be a closed set in $\mathbb{R}^k$. Let S be the set of linear combinations $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$ obtained as $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ ranges over all points of $\mathcal{A}$. Show that S is a closed subset of $\mathbb{R}^n$.

4B. Let K and L be fields, and let $K \times L$ be the product ring, with addition and multiplication defined componentwise. Find all prime ideals of $K \times L$.

5B. Let $f(z)$ be an entire function and let $a_1, \ldots, a_n$ be all zeros of f in $\mathbb{C}$. Suppose that there exist real numbers $R > 0$ and $a > 1$ such that $|f(z)| \geq |z|^a$ for all $|z| \geq R$. Prove that

$$\sum_{j=1}^{n} \mathrm{Res}_{z=a_j} \frac{1}{f(z)} = 0.$$

6B. Given a positive integer n, what are the possible values of the triple $(\mathrm{rk}(A), \mathrm{rk}(B), \mathrm{rk}(C))$ as A, B, C range over real $n \times n$ matrices satisfying $A + B + C = 0$?

7B. Let f be continuous on $[0, \infty)$ and suppose that $\lim_{x \to \infty} f(x)$ exists and is finite. Must f be uniformly continuous? Give a proof or a counterexample.

8B. Show that for every positive integer n, there exists an irreducible polynomial over $\mathbb{Q}$ of degree n such that all its roots are real.

9B. Let f be holomorphic on a neighborhood of the closed disk $\overline{B_1(0)} = \{z : |z| \leq 1\}$. Suppose that $\max_{|z|=1} |f(z)| \leq 1$. Prove that there exists a complex number z such that $|z| \leq 1$ and $f(z) = z$.