Homework 8

1. Prove that if $f \in L^1(\mathbb{R}^n)$ then $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.

Solution: Note that we have

$$\begin{aligned}\hat{f}(y+h) - \hat{f}(y) &= \int_{\mathbb{R}^n} f(x)\left(e^{-2\pi i x\cdot(y+h)} - e^{-2\pi i x\cdot y}\right) dx \\ &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot y}(e^{-2\pi i x\cdot h} - 1) dx\end{aligned}$$

Thus we have that

$$\left|\hat{f}(y+h) - \hat{f}(y)\right| \leq \int_{\mathbb{R}^n} |f(x)| \left|e^{-2\pi i x\cdot h} - 1\right| dx$$

It suffices to show that this last expression can be made arbitrarily small as we let $h \to 0$, independently of y. The idea is to apply the Dominated Convergence Theorem. Set $g_h(x) = f(x)e^{-2\pi i x\cdot(y+h)}$. Then we have that $|g_h| \leq |f|$, and $g_h \to g_0 = f(x)e^{-2\pi i x\cdot y}$ almost everywhere in $\mathbb{R}^n$. So we then have that

$$\hat{f}(y+h) = \int_{\mathbb{R}^n} g_h(x) dx \to \int_{\mathbb{R}^n} g_0(x) dx = \hat{f}(y)$$

as $h \to 0$ by Dominated Convergence.

2. Give $f \in L^2(\mathbb{R}^n)$, prove that

$$\xi \to \int_{|x|\leq N} f(x) e^{-2\pi i x\cdot\xi} dx$$

converges to $\hat{f}$ in $L^2(\mathbb{R}^n)$ as $N \to \infty$.

Solution: Let

$$g_N(\xi) = \int_{|x|\leq N} f(x) e^{-2\pi i x\cdot\xi} dx.$$

Observe that

$$\hat{f}(\xi) - g_N(\xi) = \int_{\mathbb{R}^n} f(x) 1_{\{x:|x|>N\}} e^{-2\pi i x\cdot\xi} dx = \widehat{f 1_{\{x:|x|>N\}}}(\xi).$$

Then we have that

$$\begin{aligned}\left\|\hat{f} - g_N\right\|_{L^2(\mathbb{R}^n)} &= \left\|\widehat{f 1_{\{x:|x|>N\}}}\right\|_{L^2(\mathbb{R}^n)} \\ &= \left\|f 1_{\{x:|x|>N\}}\right\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

But, for N large enough, we have that this last expression can be made smaller than any given $\epsilon > 0$ since $f \in L^2(\mathbb{R}^n)$.

3. If $f_k, f \in \mathcal{S}(\mathbb{R}^n)$ and $f_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$, then $\hat{f}_k \to \hat{f}$ and $\check{f}_k \to \check{f}$ in $\mathcal{S}(\mathbb{R}^n)$.

Solution: Recall that $f_k \to f \in \mathcal{S}(\mathbb{R}^n)$ if for all multi-indices α and β we have

$$\rho_{\alpha,\beta}(f_k - f) = \sup_{x \in \mathbb{R}^n} \left| x^\alpha (\partial^\beta (f_k - f)) \right| \to 0$$

as $k \to \infty$. We then have that

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta (\hat{f}_k - \hat{f}) \right| = C(\alpha, \beta) \sup_{x \in \mathbb{R}^n} \left| \partial^\alpha (\widehat{x^\beta (f_k - f)}) \right| \leq \left\| \partial^\alpha (x^\beta (f_k - f)) \right\|_{L^1(\mathbb{R}^n)}$$

Recall that if $f_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$, then we have that $f_k \to f$ in $L^p(\mathbb{R}^n)$ for any $0 < p \leq \infty$. Moreover, we have

$$\left\| \partial^\beta g \right\|_{L^p(\mathbb{R}^n)} \leq C(p, n) \sum_{|\alpha| = \lfloor \frac{n+1}{p} \rfloor + 1} \rho_{\alpha,\beta}(g).$$

Apply this estimate with $g = f_k - f$ and $p = 1$ to conclude that $\hat{f}_k \to \hat{f}$. Similar computations prove the statement for $\check{f}_k \to \check{f}$.

4. Find the set of eigenvalues of the Fourier transform, namely the λ such that

$$\hat{f} = \lambda f.$$

Hint: Apply the Fourier transform to the above identity, and consider functions of the form $xe^{-\pi x^2}$, $(a + bx^2)e^{-\pi x^2}$ *and* $(cx + dx^3)e^{-\pi x^2}$ *for good choices of* a, b, c, d.

Solution: Note that we have

$$\hat{\hat{f}}(x) = f(-x)$$

and so $\hat{\hat{\hat{\hat{f}}}}(x) = f(x)$. If f is an eigenfunction then we see that the corresponding eigenvalue must satisfy

$$\lambda^4 - 1 = 0.$$

From this we see that the eigenvalues of the Fourier transform are $1, -1, i, -i$. Using the remaining part of the hint, one can deduce the corresponding eigenfunctions to be Hermite polynomials.

5. If $0 < c < \infty$, define $f_c(x) = e^{-cx^2}$

   (a) Compute $\hat{f}_c$ in the following way: Let $\varphi = \hat{f}_c$ and show that $4\pi^2 t\varphi(t) + 2c\varphi'(t) = 0$ and then solve the resulting differential equation;

   (b) Show that there is one (and only one) value of c for which $f_c = \hat{f}_c$;

   (c) Show that $f_a * f_b = \gamma f_c$ where $\gamma = \gamma(a, b)$ and $c = c(a, b)$.

Solution: Part (a): Set

$$\varphi(t) = \hat{f}_c(t) = \int_{\mathbb{R}} e^{-cx^2} e^{-2\pi i x t} dx.$$

Then we have

$$\begin{aligned} \varphi'(t) &= 2\pi i \int_{\mathbb{R}} x e^{-cx^2} e^{-2\pi i x t} dx \\ &= -\frac{\pi}{c} i \int_{\mathbb{R}} \frac{d}{dx}\left(e^{-cx^2}\right) e^{-2\pi i x t} dx \\ &= -\frac{\pi}{c} i \int_{\mathbb{R}} e^{-cx^2} \frac{d}{dx}\left(e^{-2\pi i x t}\right) dx \\ &= -2\frac{\pi^2}{c} t\varphi(t). \end{aligned}$$

Rearrangement gives the resulting differential equation. Interchange of the derivative with respect to t, and switching the derivative with respect to x is justified since e^{-cx^2} is a Schwarz class function. Solving the resulting differential equation gives that

$$\hat{f}_c(t) = k e^{-\frac{\pi^2}{c} t^2}$$

where k is some constant. Note that we have that $\hat{f}_c(0) = k = \int_{\mathbb{R}} e^{-cx^2} dx = \sqrt{\frac{\pi}{c}}$. Thus we have that

$$\hat{f}_c(t) = \sqrt{\frac{\pi}{c}} e^{-\frac{\pi^2}{c} t}.$$

Part (b): Suppose that we have a value of c such that

$$f_c(t) = \hat{f}_c(t).$$

Then for all $t \in \mathbb{R}$ we have that

$$\sqrt{\frac{\pi}{c}} e^{t^2\left(\frac{\pi^2}{c} - c\right)} = 1.$$

In particular, it must be true when $t = 0$, and we obtain that $c = \pi$ is the only value that works.

Part (c): Taking the Fourier Transform, we have that

$$\widehat{f_a * f_b} = \hat{f}_a \hat{f}_b = \frac{\pi}{\sqrt{ab}} e^{-\frac{\pi^2}{a} t^2} e^{-\frac{\pi^2}{b} t^2} = \frac{\pi}{\sqrt{ab}} e^{-\pi^2\left(\frac{1}{a} + \frac{1}{b}\right) t^2} = \frac{\pi}{\sqrt{ab}} e^{-\pi^2 \frac{a+b}{ab} t^2}.$$

Set $c = c(a,b) = \frac{ab}{a+b}$ and $\gamma = \gamma(a,b) = \sqrt{\frac{\pi}{a+b}}$, then we have that

$$\frac{\pi}{\sqrt{ab}} e^{-\pi^2 \frac{a+b}{ab} t^2} = \gamma \sqrt{\frac{\pi}{c}} e^{-\frac{\pi^2}{c} t^2} = \gamma \hat{f}_c(t).$$

So we have that when $c = c(a,b) = \frac{ab}{a+b}$ and $\gamma = \gamma(a,b) = \sqrt{\frac{\pi}{a+b}}$ that

$$f_a * f_b = \gamma f_c.$$

6. Suppose that $f \in L^1(\mathbb{R}^n)$ and $f > 0$. Show that $\left|\hat{f}(y)\right| < \hat{f}(0)$ for $y \neq 0$.

Solution: Note that for any $y \in \mathbb{R}^n$ we have

$$\left|\hat{f}(y)\right| \leq \|f\|_{L^1} = \int_{\mathbb{R}^n} f(x)dx = \hat{f}(0).$$

Suppose that for some $y \neq 0$ we have that

$$\left|\hat{f}(y)\right| = \hat{f}(0).$$

This then leads to a contradiction. Indeed, we have that

$$\int_{\mathbb{R}^n} f(x)dx = \hat{f}(0) = \left|\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot y}dx\right| = \eta \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot y}dx$$

where η is a constant complex number with $|\eta| = 1$. Re-arrangement of this inequality gives that

$$\int_{\mathbb{R}^n} f(x)(1 - \eta e^{-2\pi i x\cdot y})dx = 0.$$

This implies that $f(x)(1-\eta e^{-2\pi i x\cdot y}) = 0$ almost everywhere on $\mathbb{R}^n$. But, since $y \neq 0$, we have that $(1 - \eta e^{-2\pi i x\cdot y}) \neq 0$ almost everywhere on $\mathbb{R}^n$, and so $f = 0$ almost everywhere on $\mathbb{R}^n$. This is a contradiction to the conditions on f, and so there can not be a $y \neq 0$ with equality holding. Thus we must have for $y \neq 0$ that

$$\left|\hat{f}(y)\right| < \hat{f}(0).$$

7. Compute the Fourier transform of $g(x) = e^{-2\pi|x|}$ using the following steps:

(a) Let $f \in L^1(\mathbb{R})$ and show that

$$\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f\left(x - \frac{1}{x}\right)dx.$$

(b) Use part (a) with $f(x) = e^{-tx^2}$ and $t > 0$ to obtain the following identity:

$$e^{-2t} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y-\frac{t^2}{y}} \frac{dy}{\sqrt{y}};$$

(c) Set $t = \pi |x|$ and integrate with respect to $e^{-2\pi i\xi\cdot x}dx$ to obtain that

$$\widehat{g}(\xi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.$$

Solution: Part (a): For one proof, one can check the identity when $f(x) = 1_{[0,1]}(x)$, and then use that simple functions are dense in $L^1(\mathbb{R})$. Here is another proof that one can give. Note that we have

$$\int_{\mathbb{R}} f(x)dx = \int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx.$$

We work with each of these integrands separately. For the first one, we have that

$$\int_0^\infty f(x)dx = \int_1^\infty f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)dx.$$

While for the second one, we have

$$\int_{-\infty}^0 f(x)dx = \int_{-\infty}^{-1} f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)dx.$$

Now, note that we have

$$\int_{\mathbb{R}} f\left(x - \frac{1}{x}\right)dx = \int_{-\infty}^{-1} f\left(x - \frac{1}{x}\right)dx + \int_{-1}^{1} f\left(x - \frac{1}{x}\right)dx + \int_1^\infty f\left(x - \frac{1}{x}\right)dx$$

Using these identities from above, it is easy to see that

$$\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f\left(x - \frac{1}{x}\right)dx + \int_{|x|>1} f\left(x - \frac{1}{x}\right)\frac{dx}{x^2} - \int_{-1}^{1} f\left(x - \frac{1}{x}\right)dx.$$

To conclude the computation, we are left with showing that

$$\int_{|x|>1} f\left(x - \frac{1}{x}\right)\frac{dx}{x^2} = \int_{-1}^{1} f\left(x - \frac{1}{x}\right)dx.$$

To prove this last identity one shows that

$$\int_0^1 f\left(x - \frac{1}{x}\right) = \int_{-\infty}^{-1} f\left(x - \frac{1}{x}\right)\frac{dx}{x^2}$$

and

$$\int_{-1}^0 f\left(x - \frac{1}{x}\right) = \int_1^\infty f\left(x - \frac{1}{x}\right)\frac{dx}{x^2}$$

via a standard change of variables. Part (b): We use part (a) applied to the function $f(x) = e^{-tx^2}$. Now observe that

$$\int_{\mathbb{R}} e^{-tx^2}dx = \sqrt{\frac{\pi}{t}}.$$

But, we also have that

$$\begin{aligned}
\int_{\mathbb{R}} e^{-t\left(x-\frac{1}{x}\right)^2}dx &= e^{2t}\int_{\mathbb{R}} e^{-tx^2-t\frac{1}{x^2}}dx \\
&= \frac{e^{2t}}{\sqrt{t}}\int_{\mathbb{R}} e^{-y^2-\frac{t^2}{y^2}}dy \\
&= \frac{e^{2t}}{\sqrt{t}}\int_0^\infty e^{-u-\frac{t^2}{u}}\frac{du}{\sqrt{u}}.
\end{aligned}$$

Using Part (a), we have that

$$\sqrt{\frac{\pi}{t}} = \frac{e^{2t}}{\sqrt{2t}} \int_0^\infty e^{-u-\frac{t^2}{u}} \frac{du}{\sqrt{u}}.$$

Rearrangement gives the result.

Part (c): Now set $t = \pi |x|$ in Part (b) and obtain,

$$e^{-2\pi|x|} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y-\frac{\pi^2|x|^2}{y}} \frac{dy}{\sqrt{y}}.$$

Then integrate this expression with respect to $e^{-2\pi i x\cdot\xi}dx$ to obtain that

$$\begin{aligned}
\hat{g}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi|x|} e^{-2\pi i x\cdot\xi} dx \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y-\frac{\pi^2|x|^2}{y}} \frac{dy}{\sqrt{y}} \right) e^{-2\pi i x\cdot\xi} dx \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} \left(\int_{\mathbb{R}^n} e^{-\frac{\pi^2}{y}|x|^2} e^{-2\pi i x\cdot\xi} dx \right) \frac{dy}{\sqrt{y}} \\
&= \frac{1}{\pi^{\frac{n+1}{2}}} \int_0^\infty e^{-y(1+|\xi|^2)} y^{\frac{n-1}{2}} dy \\
&= \frac{1}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}} \int_0^\infty y^{\frac{n-1}{2}} e^{-y} dy \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.
\end{aligned}$$