Solutions to practice problems

1. Let $E \subset \mathbb{R}$ be bounded, nonempty, and suppose $\sup E \notin E$. Show that E is infinite.

If E were finite, $E = \{x_1, \dots, x_n\}$, then $\sup E$ would be the largest of $x_1, \dots, x_n$, and would belong to E.

(Or: assume $a = \sup E \notin E$, and choose $x_1 \in E$. Then $x_1 < a$, so x_1 is not an upper bound for E, so E contains $x_2 > x_1$. Again, $x_2 < a$, so x_2 is not an upper bound for E, so E contains $x_3 > x_2$. Proceeding by induction, we construct infinitely many distinct elements $x_n \in E$).

2. Let $U, V \subset \mathbb{R}^2$ be open subsets satisfying $\bar{U} = \mathbb{R}^2$, $\bar{V} = \mathbb{R}^2$. Prove that $\overline{U \cap V} = \mathbb{R}^2$. (Hint: if $E \subset X$ then $\bar{E} = X$ if and only if every non-empty open set in X has non-empty intersection with E).

Using the hint: given a non-empty open subset $G \subset \mathbb{R}^2$, $G \cap U$ is non-empty (since U is dense) and open (G and U are open); so $(G \cap U) \cap V = G \cap (U \cap V)$ is non-empty (since V is dense). So every non-empty open subset of $\mathbb{R}^2$ intersects $U \cap V$, so $U \cap V$ is dense in $\mathbb{R}^2$.

(Proof of the hint: recall $\bar{E} = X$ if and only if $\forall x \in X$, $\forall r > 0$, $N_r(x)$ intersects E. First assume every non-empty open set in X intersects E, then $\forall x \in X$, $\forall r > 0$, $N_r(x)$ is open and non-empty so $N_r(x)$ intersects E, which proves that $\bar{E} = X$. Conversely, $\bar{E} = X$ and assume $G \subset X$ is open and non-empty. Take $x \in G$: then x is interior of G, so there exists $r > 0$ such that $N_r(x) \subset G$. Since $N_r(x)$ intersects E, we deduce that G also intersects E.)

Solution without using the hint: let $x \in \mathbb{R}^2$, and let $r > 0$. We have to prove that $N_r(x)$ intersects $U \cap V$ (which shows that $x \in \overline{U \cap V}$). First, since $x \in \bar{U} = X$, we know that $N_r(x)$ intersects U; let $y \in N_r(x) \cap U$. Since U and $N_r(x)$ are open, so is $N_r(x) \cap U$, so there exists $r' > 0$ such that $N_{r'}(y) \subset N_r(x) \cap U$. Since $y \in \bar{V} = X$, we know that $N_{r'}(y)$ intersects V. Let $z \in N_{r'}(y) \cap V$. Since $N_{r'}(y) \subset N_r(x) \cap U$, we have $z \in (U \cap V) \cap N_r(x)$. Therefore $U \cap V$ intersects all neighborhoods of x, and so $x \in \overline{U \cap V}$.

3. If A and B are compact subsets of X, show that $A \cup B$ is compact.

Let $\{G_\alpha\}$ be an open cover of $A \cup B$: then $\bigcup G_\alpha \supset A$, and A is compact, so there exist $\alpha_1, \dots, \alpha_n$ such that $G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \supset A$. Similarly, there exist $\alpha'_1, \dots, \alpha'_m$ such that $G_{\alpha'_1} \cup \dots \cup G_{\alpha'_m} \supset B$. Then $G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \cup G_{\alpha'_1} \cup \dots \cup G_{\alpha'_m}$ is a finite subcover of $A \cup B$.

4. Let $\{x_n\}$ be a sequence satisfying $|x_n| \leq \frac{1}{3^n}$ for each $n \geq 1$. Put $y_n = x_1 + \dots + x_n$. Prove that the sequence $\{y_n\}$ is convergent.

The series $\sum \frac{1}{3^n}$ is convergent, so by the comparison criterion (Theorem 3.25) $\sum x_n$ is convergent.

(Or: $\{y_n\}$ is a Cauchy sequence since, for $m \geq n \geq N$, $|y_m - y_n| = |x_{n+1} + \dots + x_m| \leq \frac{1}{3^{n+1}} + \dots + \frac{1}{3^m} \leq \frac{1}{3^{n+1}}(1 + \frac{1}{3} + \frac{1}{9} + \dots) = \frac{3}{2}\frac{1}{3^{n+1}} \leq \frac{3}{2}\frac{1}{3^{N+1}}$, which can be made smaller than any $\epsilon > 0$ by taking N large enough.)

5. Find all the subsequential limits of each of the following sequences: $a_n = n \sin \frac{n\pi}{4}$; $a_n = 1 - \frac{(-1)^n}{n}$; $a_n = 1 - (-1)^n$. Are these sequences bounded? convergent?

a) Observe that $a_n = 0$ if n is a multiple of 4; $a_n = \pm n$ if $n = 4k + 2$ for some integer k; $a_n = \pm n/\sqrt{2}$ if n is odd. Therefore 0 is a subsequential limit (take $\{a_{4k}\}$), and it is the only finite subsequential limit of $\{a_n\}$ since the non-zero terms all satisfy $|a_n| \geq n/\sqrt{2}$; there are also subsequences which diverge to $+\infty$ or to $-\infty$. The sequence is not bounded, and not convergent.

b) $|a_n - 1| = \frac{1}{n} \to 0$, so $a_n \to 1$. The sequence is bounded and convergent, and all its subsequences converge to 1.

c) a_n equals 0 for even n, and 2 for odd n, so the subsequential limits are 0 and 2. The sequence is bounded but not convergent.

6. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$. Give an example to show that equality need not hold.

Let $a^* = \limsup a_n$ and $b^* = \limsup b_n$, and fix $\epsilon > 0$. Then all but finitely many terms of $\{a_n\}$ satisfy $a_n < a^* + \epsilon$, and all but finitely many terms of $\{b_n\}$ satisfy $b_n < b^* + \epsilon$ (Theorem 3.17(b)). Hence, there exists N such that $a_n + b_n < a^* + b^* + 2\epsilon$ for all $n \geq N$. This implies that $\limsup(a_n+b_n) \leq a^*+b^*+2\epsilon$. Since this holds for all $\epsilon > 0$, we must have $\limsup(a_n+b_n) \leq a^*+b^*$.

Equality need not hold: let $a_n = (-1)^n$, $b_n = -(-1)^n$, then $\limsup a_n = \limsup b_n = 1$, but $a_n + b_n = 0$ so $\limsup(a_n + b_n) = 0 < 1 + 1$.

7. Find a countable subset of $\mathbb{R}$ with (a) exactly two limit points; (b) countably many limit points; (c) uncountably many limit points.

a) $A = \{\frac{1}{n},\ n = 1, 2, \dots\} \cup \{1 + \frac{1}{n},\ n = 1, 2, \dots\}$ (the limit points are 0 and 1).

b) $A = \{\frac{1}{m} + \frac{1}{n},\ m, n = 1, 2, \dots\}$ (the limit points are 0 and all the $\frac{1}{n}$).

c) $A = \mathbb{Q}$ (all real numbers are limit points).

8. Let A, B be subsets of a metric space, and denote by A°, B° the sets of interior points of A, B. Prove that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

If $x \in (A \cap B)^\circ$ then x is an interior point of $A \cap B$, i.e. $\exists\, r > 0$ such that $N_r(x) \subset A \cap B$. Then $N_r(x) \subset A$, so $x \in A^\circ$, and $N_r(x) \subset B$, so $x \in B^\circ$. Therefore $x \in A^\circ \cap B^\circ$. This proves $(A \cap B)^\circ \subset A^\circ \cap B^\circ$. Conversely, let $x \in A^\circ \cap B^\circ$. Since x is an interior point of A, $\exists\, r_1 > 0$ such that $N_{r_1}(x) \subset A$; similarly x is an interior point of B so $\exists\, r_2 > 0$ such that $N_{r_2}(x) \subset B$. Let $r = \min\{r_1, r_2\}$. Then $N_r(x) \subset A \cap B$. So $x \in (A \cap B)^\circ$, so $A^\circ \cap B^\circ \subset (A \cap B)^\circ$.

(Or, using results seen in lecture: $A^\circ \subset A$, $B^\circ \subset B$ are open, so $A^\circ \cap B^\circ$ is open and contained in $A \cap B$, which implies that $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. Conversely, $(A \cap B)^\circ$ is open and contained in A, so it is contained in A°; similarly it is open and contained in B, so contained in B°; so $(A \cap B)^\circ \subset A^\circ \cap B^\circ$).

9. Assume that $\sum a_n$ is a convergent series and that $a_n \geq 0\ \forall n \geq N$. Prove that $\sum \frac{1}{n}\sqrt{a_n}$ converges. (Hint: consider the quantity $(\sqrt{a_n} - \frac{1}{n})^2$, and use the comparison criterion).

(Assigned on homework).

10. Give an example of a countable compact subset of $(\mathbb{R}, d)$.

$\{\frac{1}{n},\ n = 1, 2, \dots\} \cup \{0\}$ (closed and bounded, hence compact; see also Problem set 3).

11. True or false?

– if a subset $A \subset \mathbb{R}$ has a least upper bound in $\mathbb{R}$ then it also has a greatest lower bound in $\mathbb{R}$;

False. Consider e.g. $(-\infty, 0)$.

– if E is a finite subset of a metric space (X, d) then E is closed in X;

True. E has no limit points, so all limit points of E belong to E.

– if K is a compact subset of a metric space (X, d) and $F \subset X$ is closed in X, then $K \cap F$ is closed in X.

True. K is closed in X (Theorem 2.34), so $K \cap F$ is closed. (In fact $K \cap F$ is even compact, by Theorem 2.35).

12. Let E be an open subset of $\mathbb{R}^2$. Is every point of E a limit point of E? Same question if E is closed.

Let $x \in E$, then x is an interior point of E, hence there is $r_0 > 0$ such that $N_{r_0}(x) \subset E$. Hence, for all $r > 0$, $N_r(x) \cap E \supset N_r(x) \cap N_{r_0}(x) = N_{\min(r,r_0)}(x)$ contains points other than x. (Note: this need not be true in a general metric space (X, d), it could be that this neighborhood contains no other point, if x is an isolated point of X! However, in $\mathbb{R}^k$ neighborhoods are uncountable). Hence x is a limit point of E.

This property does not hold for closed E: for example $E = \{0\}$ is closed, but 0 is not a limit point of E.

13. If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + s_n}$ $(n = 1, 2, 3, \dots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for all n. (Hint: show that $\{s_n\}$ is a monotonic sequence).

First, $s_1 = \sqrt{2} < s_2 = \sqrt{2 + \sqrt{2}} < 2$. By induction we prove that $s_n < s_{n+1} < 2$ for all n: assume that $s_{n-1} < s_n < 2$, then $2 + s_{n-1} < 2 + s_n < 4$, so $\sqrt{2 + s_{n-1}} < \sqrt{2 + s_n} < 2$, i.e. $s_n < s_{n+1} < 2$. This proves that $s_n < 2$ for all n, and that $\{s_n\}$ is monotonically increasing. Since $\{s_n\}$ is monotonic and bounded, it converges.

14. Find $\limsup s_n$ and $\liminf s_n$, where $\{s_n\}$ is the sequence defined by $s_1 = 0$, $s_{2m} = \frac{s_{2m-1}}{2}$, $s_{2m+1} = \frac{1}{2} + s_{2m}$.

The first few terms are: $0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \dots$. Consider the odd terms: $s_{2m+1} = \frac{1}{2} + s_{2m} = \frac{1}{2} + \frac{1}{2}s_{2m-1} = \frac{1}{2}(1 + s_{2m-1})$. By induction, $s_{2m+1} = 1 - \frac{1}{2^m}$, and $s_{2m+1} \to 1$. Moreover, $s_{2m} = \frac{1}{2}s_{2m-1} = \frac{1}{2}(1 - \frac{1}{2^{m-1}}) = \frac{1}{2} - \frac{1}{2^m}$, and $s_{2m} \to \frac{1}{2}$. So $\liminf s_n = \frac{1}{2}$ and $\limsup s_n = 1$.

15. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_k}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Let $\epsilon > 0$. There exists N such that, for $n, m \geq N$, $d(p_n, p_m) < \epsilon$. Then, consider any $n \geq N$: for k sufficiently large (so that $n_k \geq N$), $d(p_n, p_{n_k}) < \epsilon$. Taking the limit as $k \to \infty$, it follows that $d(p_n, p) \leq \epsilon$. Or: if k is sufficiently large then $d(p_{n_k}, p) < \epsilon$ (by the assumption $p_{n_k} \to p$), so $d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < 2\epsilon$. In any case, we conclude that $d(p_n, p)$ becomes arbitrarily small for n large, i.e. $p_n \to p$.