Ring and Module Theory Qual Review

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1 (Some) qual problems

- (Spring 2007, 2) Let I, J be two ideals in a commutative ring R (with unit).
  (a) Define $K = \{r : rJ \leq I\}$. Show that K is an ideal
  (b) If R is a PID, so $I = \langle i \rangle$, $J = \langle j \rangle$, give a formula for a generator k of K.
- (Spring 2007, 3) Describe up to isomorphism all the $\mathbb{R}[x]$-module structure one might put on a 3 dimensional real vector space (extending the $\mathbb{R}$ action).

  Fundamental theorem of modules over a PID.
- (Fall 2009, 8) Let R be a commutative ring with identity. Suppose I and J are ideals of R such that R/I and R/J are noetherian rings. Prove that $R/(I \cap J)$ is also a noetherian ring.
- (Spring 2012, 3) Let R be a commutative ring.
  (a) Suppose that R is noetherian. Show that if $\varphi : R \to R$ is a surjective ring homomorphism, then it is injective.
  (b) If R is not noetherian, must a surjective ring homomorphism be injective? Prove or give a counterexample.

  item (Spring 2012, 5) Consider $f \in F[x]$ where F is an algebraically closed field. Suppose that f has the property that for all matrices $A \in M_n(F)$ of any size n, if $f(A) = 0$, then A is a diagonalizable matrix; then we say the polynomial f forces diagonalizability.
  (a) Characterize a simple rule exactly which polynomials in $F[x]$ prove diagonalizability. Prove your answer.
  (b) Fix $m \geq 1$. Is every square matrix A with entries in F satisfying$A^m = I$ diagonalizable? (The answer depends on F).
- (Fall 2009, 2) Let R denote a commutative ring and I an ideal, $I \neq R$.
  (a) Give an example where R/I has nilpotents but R doesn't.
  (b) Give an example where R has nilpotents but R/I doesn't.
- (Fall 2009, 3) Let $\varphi : \mathbb{C}[x] \to F$ be a ring homomorphism where F is a field, $\varphi(1) \neq 0$.
  (a) Give an example where φ is not onto.

(b) If φ is onto, show that $F \cong \mathbb{C}$.

- (Fall 2009, 4)
  (a) Give an example of two finitely generated $\mathbb{Z}$-modules, M and N such that M, N are not isomorphic (as $\mathbb{Z}$-modules) but $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q} \otimes_{\mathbb{Z}} N$ (as $\mathbb{Q}$-modules).
  (b) Let M be a finitely generated $\mathbb{R}[x]$-module, described using the classification of f.g. modules over a PID. Give a similar description of $\mathbb{C}[x] \otimes_{\mathbb{R}[x]} M$ as a $\mathbb{C}[x]$-module.

2 (Some) ring things to know

- Basic facts and definitions (homomorphisms, isomorphism theorems, subrings, ideals, quotient rings, etc.)
- Any finite integral domain is a field.
- Isomorphism theorems
- A commutative ring R is a field if and only if its only ideals are 0 and R.
- Kernels of ring homomorphisms are ideals.
- For a commutative ring R and an ideal I, R/I is a domain (resp. field) if and only if I is prime (resp. maximal).
- (Chinese Remainder Theorem) Let $A_1, \dots, A_k$ be ideals of R. Then map

$$R \to R/A_1 \times \cdots \times R/A_k$$

  has kernel $A_1 \cap \cdots \cap A_k$. If for each $i \neq j$, A_i and A_j are comaximal, then the map is surjective and $A_1 \cap \cdots A_k = A_1 \cdots A_k$ so

$$R/(A_1 \cdots A_k) = R/(A_1 \cap \cdots \cap A_k) \cong R/A_1 \times \cdots \times R/A_k.$$

- fields $\subsetneq$ Euclidean domains $\subsetneq$ PIDs $\subsetneq$ UFDs $\subsetneq$ Integral domains $\subsetneq$ Rings
  Examples showing strict inclusion: $F[x]$, $\mathbb{Z}[(1+\sqrt{-19})/2]$, $F[x, y]$, $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}_4$
- A Euclidean domain is a PID, an ideal is generated by an element of minimum norm.
- If $(a, b) = (d)$, where $d = \gcd(a, b)$.
- If $R[x]$ is a PID then R is a field.
- If F is a field then $F[x]$ is a Euclidean domain.
- Every prime ideal in a PID is maximal.
- In a UFD, an element is prime if and only if it is irreducible.
- R is a UFD if and only if $R[x]$ is a UFD.

- Gaussian integers
- (Gauss' Lemma) Let R be a UFD with field of fractions F and $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$ then $p(x)$ is reducible in $R[x]$.
- (Eisenstein's criterion) Let P be a prime ideal of an integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_x + a_0$ be a polynomial in $R[x]$. Suppose $a_{n-1}, \ldots, a_1, a_0 \in P$ and $a_0 \notin P^2$. Then $f(x)$ is irreducible.

3 (Some) module things to know

- Basic facts and definitions (homomorphisms, isomorphism theorems, submodules, quotient ideals, direct products, cyclic modules)
- An $F[x]$-module V is an F-vector space with a linear transformation $V \to V$.
- Free modules (Let A be any set and consider the free module $F(A)$. If M is any R-module, $\varphi A \to M$ any homomorphism, this lifts to a unique R-module homomorphism $\Phi : F(A) \to M$.)
- Tensor product (commutative, associative, right exact, splits over direct sums)
- (Universal property) Any R-bilinear map $M \times N \to L$ induces a unique R-module homomorphism $M \otimes_R N \to L$.
- Extension of scalars
- Hom (is left exact)
- Exact sequences (short, split)
- Projective modules (direct summand of a free module, lifting property, every short exact sequence ending with a projective splits)

  (Lifting property). Given a surjection $M \to N$ and any homomorphism $\varphi P \to N$, φ lifts to a (not necessarily unique) homomorphism $P \to N$.
- Injective modules (lifting property, every short exact sequence starting with an injective splits)

  (Lifting property dual to the projective property). For any injection $L \to M$, and any homomorphism $\varphi L \to Q$, φ lifts to a homomorphism $Q \to M$.
- (Baer's criterion) An R-module Q is injective if and only if for every left ideal I, any module homomorphism $I \to Q$ can be extended to one $R \to Q$.
- If R is a PID, Q is injective if and only if $rQ = Q$ for all $0 \neq r \in R$.
- Flat modules (tensoring with a flat module is exact)
- M is a noetherian R-module if nad only if every nonempty set of submodules of M contains a maximal element if and only if every submodule of$M$ is finitely generated.

- (Fundamental Theorem of Modules over a PID)

  Invariant factor form

  Let R be a PID and M a finitely gnereated R-module. Then

  $$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$$

  for some $r \in \mathbb{Z}_{\geq 0}$ and $a_1 \mid a_2 \mid \cdots \mid a_m$.

  Rational canonical form

  Elementary divisor form:

  $$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus \cdots R/(p_t^{\alpha_t})$$

  for some $r \in \mathbb{Z}_{\geq 0}$ and $p_1^{\alpha_1}, \ldots, p_t^{\alpha_t}$ are positive powers of not necessarily distinct primes.

  Jordan canonical form

- Characteristic polynomials, minimal polynomials