1A. Consider a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$ with the property that for each $x \in [a,b]$ there is an open interval I_x containing x such that $(f_n)_{n\geq 1}$ converges uniformly in $I_x \cap [a,b]$. Show that $(f_n)_{n\geq 1}$ converges uniformly in $[a,b]$.

2A. Find a countable abelian group whose endomorphism ring has the same cardinality as the set of real numbers. Justify your answer.

3A. Let $a_1,\ldots,a_n,b_1,\ldots,b_m$ be distinct complex numbers, let $r_1,\ldots,r_n$ be nonnegative integers, and let $c_1,\ldots,c_m$ be complex numbers. Prove that if $m \leq r_1+\cdots+r_n+1$, then there exists a rational function $F(z) \in \mathbb{C}(z)$ satisfying all of the following:

1. $F(z)$ is holomorphic at ∞ and everywhere in $\mathbb{C}$ except possibly at $a_1,\ldots,a_n$.
2. $\operatorname{ord}_{z=a_i} F(z) \geq -r_i$
3. $F(b_j) = c_j$ for $j = 1,\ldots,m$.

4A. For which positive integers n is it true that every invertible 2×2 matrix A with real entries can be expressed as the n-th power of another 2×2 matrix with real entries?

5A. Suppose $f\colon \mathbb{R} \to \mathbb{C}$ satisfies $f'(t) + 2itf(t) = e^{2it}$ and $f(0) = 0$. Compute

$$\lim_{t\to+\infty} e^{it^2}(f(t) - f(-t)).$$

You may assume $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$.

6A. For which pairs of integers (a,b) is the quotient ring $\mathbb{Z}[x]/(x^2+ax+b)$ isomorphic (as a ring) to the direct product of rings $\mathbb{Z}\times\mathbb{Z}$?

7A. Evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

8A. Let V and W be finite-dimensional vector spaces over a field k. Let $f\colon V^n \to W$ be a function such that

(a) For each fixed $i \in \{1,\ldots,n\}$ and fixed $v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n \in V$, the map
$$\begin{aligned} V &\to W \\ x &\mapsto f(v_1,\ldots,v_{i-1},x,v_{i+1},\ldots,v_n) \end{aligned}$$
is a k-linear transformation; and

(b) $f(v_1,\ldots,v_n) = 0$ whenever $v_i = v_{i+1}$ for some $i \in \{1,\ldots,n-1\}$.

Prove that either $\dim V \geq n$ or f is identically zero.

9A. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function, and let L be a nonnegative real number. Prove that the following are equivalent:

(i) For every $x,y \in \mathbb{R}^n$,
$$(f(x)-f(y)).(x-y) \leq L|x-y|^2$$

(ii) For every $x,v \in \mathbb{R}^n$,
$$Df(x)v.v \leq L|v|^2,$$

where $Df(x)$ is the derivative of f at x, and . denotes the standard inner product of vectors in $\mathbb{R}^n$.

1B. Let F be a field (of arbitrary characteristic). Suppose g is a nonnegative integer, and polynomials $a(x), b(x) \in F[x]$ satisfy $\deg a(x) \leq g$ and $\deg b(x) = 2g + 1$. Prove that the polynomial $y^2 + a(x)y + b(x)$ is irreducible over $F(x)$.

2B. Find the maximum possible value of $|f'(1)|$ given that f is holomorphic on an open neighborhood of $\{z \in \mathbb{C} : |z| \leq 2\}$ and satisfies $|f(z)| \leq 1$ when $|z| = 2$.

3B. Let A be a $d \times d$ matrix with complex entries. Assume that every eigenvalue of A has absolute value 1. Prove that there exists a constant $c \in \mathbb{R}$ independent of n such that

$$\|A^n x\| \leq c n^{d-1} \|x\|$$

for all $n \geq 1$ and $x \in \mathbb{C}^d$. Here $\|x\| := (|x_1|^2 + \cdots + |x_d|^2)^{1/2}$ for all $(x_1, \ldots, x_d) \in \mathbb{C}^d$.

4B. Let $a_1, \ldots, a_n$ be positive real numbers. Let Δ be the set of points $\mathbf{x} \in \mathbb{R}^n$ satisfying the conditions

$$\sum_{i=1}^{n} a_i x_i = 1, \quad x_i > 0 \text{ for all } i.$$

Prove that the function $\log(\prod_{i=1}^{n} x_i)$ has a unique maximum on Δ and find the point where it occurs.

5B. Let $n_1, \ldots, n_r$ be integers ≥ 2. Prove that there is a finite group G containing elements $g_1, \ldots, g_r$ such that g_i has exact order n_i for each i, and $g_i g_j \neq g_j g_i$ for $i \neq j$.

6B. Let $(u_n(x, y))_{n \geq 1}$ be a sequence of functions that are defined and harmonic for (x, y) in an open neighborhood of the upper half plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$. Suppose that $\frac{\partial u_n}{\partial y}(x, 0) = 0$ for all $x \in \mathbb{R}$, and $u_n(x, 0)$ converges to 0 as $n \to \infty$ uniformly for $x \in \mathbb{R}$. Must $u_n(x, y) \to 0$ as $n \to \infty$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}_{>0}$?

7B. Let A and B be $n \times n$ matrices with complex entries, such that $AB - BA$ is a linear combination of A and B. Prove that there exists a nonzero vector v that is an eigenvector of both A and B.

8B. For each real number x, compute

$$\lim_{n \to \infty} n \left(\left(1 + \frac{x}{n}\right)^n - e^x \right).$$

9B. Let S_4 be the group of permutations of $\{1, 2, 3, 4\}$. Determine the order of the automorphism group $\mathrm{Aut}(S_4)$. Justify your answer.