School of Mathematics and Statistics

MT5824 Topics in Groups

Problem Sheet VII: Nilpotent groups (Solutions)

1. **Show that $\gamma_2(G) = G'$. Deduce that abelian groups are nilpotent.**

Solution: By definition $\gamma_2(G) = [G, G]$. Thus $\gamma_2(G) = \langle\, [x, y] \mid x, y \in G \,\rangle = G'$. If G is abelian, then $[x, y] = 1$ for all $x, y \in G$, so $\gamma_2(G) = G' = \mathbf{1}$. Hence G is nilpotent (of class $\leqslant 1$).

2. **Show that $\mathrm{Z}(S_3) = 1$. Hence calculate the upper central series of S_3 and deduce that S_3 is not nilpotent.**

Show that $\gamma_i(S_3) = A_3$ for all $i \geqslant 2$. [Hint: We have calculated S_3' previously and now know that S_3 is not nilpotent.]

Find a normal subgroup N of S_3 such that S_3/N and N are both nilpotent.

Solution: Recall that all permutations with the same cycle structure are conjugate in S_n. Therefore a permutation lies in the centre of S_3 if and only if it is the only permutation of its cycle structure. Hence $\mathrm{Z}(S_3) = \mathbf{1}$ (there are three permutations of cycle structure $(\alpha\,\beta)$ and two of cycle structure $(\alpha\,\beta\,\gamma)$).

This shows that $\mathrm{Z}_1(S_3) = \mathbf{1}$. Suppose that $\mathrm{Z}_i(S_3) = \mathbf{1}$. Then $\mathrm{Z}_{i+1}(S_3) = \mathrm{Z}_{i+1}(S_3)/\mathrm{Z}_i(S_3) = \mathrm{Z}(S_3/\mathrm{Z}_i(S_3)) = \mathrm{Z}(S_3) = \mathbf{1}$. Hence, by induction, $\mathrm{Z}_i(S_3) = \mathbf{1}$ for all i. Since $\mathrm{Z}_i(S_3) < S_3$ for all i, we deduce that S_3 is not nilpotent.

Now $S_3' = A_3$, by Question 2(i) on Problem Sheet VI. Hence $\gamma_2(S_3) = S_3' = A_3$. Now A_3 is of order 3, so has no proper non-trivial subgroups. Hence for $i > 2$, either $\gamma_i(S_3) = A_3$ or $\gamma_i(S_3) = \mathbf{1}$. But S_3 is not nilpotent, so $\gamma_i(S_3) \neq \mathbf{1}$ for all i. Hence $\gamma_i(S_3) = A_3$ for all $i \geqslant 2$.

Let $N = A_3 \trianglelefteq S_3$. Then $S_3/N \cong C_2$ and $N \cong C_3$, so these are both abelian and hence nilpotent. (Thus we have an example of a non-nilpotent group G with normal subgroup N such that G/N and N are nilpotent.)

3. **Show that $\mathrm{Z}(G \times H) = \mathrm{Z}(G) \times \mathrm{Z}(H)$.**

Show, by induction on i, that $\mathrm{Z}_i(G \times H) = \mathrm{Z}_i(G) \times \mathrm{Z}_i(H)$ for all i.

Deduce that a direct product of a finite number of nilpotent groups is nilpotent.

Solution: Let $(x, y) \in \mathrm{Z}(G \times H)$. Then for $g \in G$ and $h \in H$, it follows that $(x, y)(g, h) = (g, h)(x, y)$. That is, $(xg, yh) = (gx, hy)$. Hence $xg = gx$ for all $g \in G$, and $yh = hy$ for all $h \in H$. Therefore $x \in \mathrm{Z}(G)$ and $y \in \mathrm{Z}(H)$, so $\mathrm{Z}(G \times H) \leqslant \mathrm{Z}(G) \times \mathrm{Z}(H)$.

Conversely, if $(x, y) \in \mathrm{Z}(G) \times \mathrm{Z}(H)$; that is, $x \in \mathrm{Z}(G)$ and $y \in \mathrm{Z}(H)$, then

$$(x, y)(g, h) = (xg, yh) = (gx, hy) = (g, h)(x, y)$$

so $(x, y) \in \mathrm{Z}(G \times H)$. This shows that $\mathrm{Z}(G) \times \mathrm{Z}(H) \leqslant \mathrm{Z}(G \times H)$. The equality now follows.

For the next step, induct on i. If $i = 0$, then $\mathrm{Z}_0(G \times H) = \{(1,1)\} = \mathbf{1} \times \mathbf{1} = \mathrm{Z}_0(G) \times \mathrm{Z}_0(H)$, so the result holds. Suppose as an inductive hypothesis that $\mathrm{Z}_i(G \times H) = \mathrm{Z}_i(G) \times \mathrm{Z}_i(H)$ for some i. Then

$$\frac{G \times H}{\mathrm{Z}_i(G \times H)} = \frac{G \times H}{\mathrm{Z}_i(G) \times \mathrm{Z}_i(H)}.$$

The map ϕ that sends $(\mathrm{Z}_i(G) \times \mathrm{Z}_i(H))(x,y)$ to $(\mathrm{Z}_i(G)x, \mathrm{Z}_i(H)y)$ is an isomorphism:

$$\phi \colon \frac{G \times H}{\mathrm{Z}_i(G) \times \mathrm{Z}_i(H)} \to \frac{G}{\mathrm{Z}_i(G)} \times \frac{H}{\mathrm{Z}_i(H)}.$$

(This works whenever $M \trianglelefteqslant G$ and $N \trianglelefteqslant G$, for then $(G \times H)/(M \times N) \cong G/M \times H/N$ via a similar isomorphism.) This isomorphism ϕ maps the centre of the group on the left-hand side to the centre of the group on the right-hand side. Hence

$$\begin{aligned}
\left(\frac{\mathrm{Z}_{i+1}(G \times H)}{\mathrm{Z}_i(G \times H)}\right)\phi &= \left(\mathrm{Z}\left(\frac{G \times H}{\mathrm{Z}_i(G \times H)}\right)\right)\phi \\
&= \left(\mathrm{Z}\left(\frac{G \times H}{\mathrm{Z}_i(G) \times \mathrm{Z}_i(H)}\right)\right)\phi \\
&= \mathrm{Z}(G/\mathrm{Z}_i(G) \times H/\mathrm{Z}_i(G)) \\
&= \mathrm{Z}(G/\mathrm{Z}_i(G)) \times \mathrm{Z}(H/\mathrm{Z}_i(G)) && \text{from the first part} \\
&= \mathrm{Z}_{i+1}(G)/\mathrm{Z}_i(G) \times \mathrm{Z}_{i+1}(H)/\mathrm{Z}_i(H) && \text{by definition} \\
&= \left(\frac{\mathrm{Z}_{i+1}(G) \times \mathrm{Z}_{i+1}(H)}{\mathrm{Z}_i(G) \times \mathrm{Z}_i(H)}\right)\phi
\end{aligned}$$

with the last step being the definition of ϕ. Since ϕ is a bijection,

$$\frac{\mathrm{Z}_{i+1}(G \times H)}{\mathrm{Z}_i(G \times H)} = \frac{\mathrm{Z}_{i+1}(G) \times \mathrm{Z}_{i+1}(H)}{\mathrm{Z}_i(G \times H)}$$

and the Correspondence Theorem yields $\mathrm{Z}_{i+1}(G \times H) = \mathrm{Z}_{i+1}(G) \times \mathrm{Z}_{i+1}(H)$, which completes the induction.

Let $G_1, G_2, \ldots, G_n$ be nilpotent groups. Then there exist c_i such that $\mathrm{Z}_{c_i}(G_i) = G_i$. Choose c to be the largest of all the c_i. Then $\mathrm{Z}_c(G_i) = G_i$ for $i = 1, 2, \ldots, n$. By the previous result, we see that

$$\begin{aligned}
\mathrm{Z}_c(G_1 \times G_2 \times \cdots \times G_n) &= \mathrm{Z}_c(G_1) \times \mathrm{Z}_c(G_2) \times \cdots \times \mathrm{Z}_c(G_n) \\
&= G_1 \times G_2 \times \cdots \times G_n,
\end{aligned}$$

and hence $G_1 \times G_2 \times \cdots \times G_n$ is nilpotent.

4. Let G be an finite elementary abelian p-group. Show that $\Phi(G) = 1$.

Solution: Let $G = C_p \times C_p \times \cdots \times C_p$ (d times, for some d). Then

$$M = M_i = C_p \times \cdots \times C_p \times \mathbf{1} \times C_p \times \cdots \times C_p$$

(where the $\mathbf{1}$ occurs in the ith entry) is a subgroup of G of index p. If H is a subgroup of G such that $M \leqslant H \leqslant G$, then $|G : H| \cdot |H : M| = |G : M| = p$, so as p is prime, either $H = G$ or $H = M$. Hence M is a maximal subgroup of G. Clearly

$$\bigcap_{i=1}^{d} M_i = \mathbf{1}$$

and this is the intersection of just some of the maximal subgroups of G. Hence

$$\Phi(G) = \bigcap_{\substack{M \text{ maximal} \\ \text{in } G}} M \leqslant \bigcap_{i=1}^{d} M_i = \mathbf{1}.$$

5. **Let G be a finite p-group.**
If M is a maximal subgroup of G, show that $|G : M| = p$. [Hint: G is nilpotent, so $M \trianglelefteq G$.]
Deduce that $G^p G' \leqslant \Phi(G)$.
Use the previous question to show that $\Phi(G) = G^p G'$.

Show that G can be generated by precisely d elements if and only if $G/\Phi(G)$ is a direct product of d copies of the cyclic group of order p.

Solution: Since G is a finite p-group, it is nilpotent (Example 7.6). Let M be a maximal subgroup of G. Then $M \trianglelefteq G$ (Lemma 7.15), and G/M possesses no non-trivial proper subgroups (by the Correspondence Theorem). Therefore G/M is cyclic of prime order, so $|G : M| = p$.

If $x \in G$, then $(Mx)^p = M1$, so $x^p \in M$. Hence

$$x^p \in \bigcap_{M \text{ maximal}} M = \Phi(G) \qquad \text{for all } x \in G.$$

We deduce that $G^p = \langle\, x^p \mid x \in G \,\rangle \leqslant \Phi(G)$. We have already observed that $G' \trianglelefteq \Phi(G)$ (see Theorem 7.18), so

$$G^p G' \leqslant \Phi(G).$$

Let $N = G^p G'$. This is a product of two normal subgroups of G, so $N \trianglelefteq G$. Now G/N is abelian (since $G' \leqslant N$) and if $x \in G$, then

$$(Nx)^p = Nx^p = N1$$

(since $x^p \in G^p \leqslant N$). Hence G/N is an elementary abelian p-group. It is therefore a direct product of a number of copies of C_p. The previous question now gives $\Phi(G/N) = \mathbf{1}$. Hence there is a collection M_1, M_2, $\dots$, M_k of subgroups of G containing N such that M_i/N is a maximal subgroup of G/N and $\bigcap_{i=1}^{k}(M_i/N) = \mathbf{1}$. By the Correspondence Theorem, M_i is a maximal subgroup of G and

$$\bigcap_{i=1}^{k} M_i = N.$$

Hence

$$\Phi(G) = \bigcap_{\substack{M \text{ maximal} \\ \text{in } G}} M \leqslant \bigcap_{i=1}^{k} M_i = N = G^p G'.$$

Taken together with the previous inclusion, $\Phi(G) = G^p G'$.

Now as $G/\Phi(G)$ is an elementary abelian p-group, it is a direct product of d copies of the cyclic group C_p (for some d). Choose x_1, x_2, $\dots$, $x_d \in G$ such that

$$\Phi(G)x_1,\ \Phi(G)x_2, \dots,\ \Phi(G)x_d$$

are the generators of these d direct factors. If $g \in G$, then

$$\Phi(G)g = \Phi(G)x_1^{e_1} x_2^{e_2} \dots x_d^{e_d}$$

for some $e_i \in \{0, 1, \dots, p-1\}$, so $g = y x_1^{e_1} x_2^{e_2} \dots x_d^{e_d}$ where $y \in \Phi(G)$. Hence

$$G = \langle x_1, x_2, \dots, x_d, \Phi(G) \rangle.$$

Suppose that $x_1, x_2, \dots, x_d$ do not generate G. Then $\langle x_1, x_2, \dots, x_d \rangle$ is a proper subgroup of G, so there exists a maximal subgroup M such that

$$\langle x_1, x_2, \dots, x_d \rangle \leqslant M < G.$$

Then $x_1, x_2, \dots, x_d \in M$ while, by definition, $\Phi(G) \leqslant M$. Hence

$$G = \langle x_1, x_2, \dots, x_d, \Phi(G) \rangle \leqslant M < G,$$

a contradiction. So $x_1, x_2, \dots, x_d$ generate G. This shows that if $G/\Phi(G)$ is a direct product of d copies of C_p, then G can be generated by d elements.

On the other hand, if G can be generated by d elements, then so can every quotient. A direct product of more than d copies of C_p cannot be generated by d elements, so the number of copies of C_p appearing in the direct product for $G/\Phi(G)$ is at most d.

Putting the above together we deduce that G can be generated by precisely d elements (and no fewer) if and only if $G/\Phi(G)$ is a direct product of d copies of the cyclic group C_p of order p.

6. Let G be a nilpotent group with lower central series

$$G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_c(G) > \gamma_{c+1}(G) = 1.$$

Suppose N is a non-trivial normal subgroup of G. Choose i to be the largest positive integer such that $N \cap \gamma_i(G) \neq 1$. Show that $[N \cap \gamma_i(G), G] = 1$.

Deduce that $N \cap \mathrm{Z}(G) \neq 1$.

Solution: $N \neq \mathbf{1}$, so $N \cap \gamma_1(G) = N \cap G = N \neq \mathbf{1}$. Hence we may choose i to be the largest positive integer such that $N \cap \gamma_i(G) \neq \mathbf{1}$. Then

$$[N \cap \gamma_i(G), G] \leqslant [\gamma_i(G), G] = \gamma_{i+1}(G)$$

while

$$[N \cap \gamma_i(G), G] \leqslant [N, G] \leqslant N$$

since $N \trianglelefteq G$ (for if $x \in N$ and $g \in G$, then $[x, g] = x^{-1} x^g \in N$). Hence

$$[N \cap \gamma_i(G), G] \leqslant N \cap \gamma_{i+1}(G) = \mathbf{1}$$

by the hypothesis that i is largest with the given property.

Hence $N \cap \gamma_i(G) \leqslant \mathrm{Z}(G)$ since $[x, g] = 1$ for all $x \in N \cap \gamma_i(G)$ and all $g \in G$. Therefore

$$\mathbf{1} \neq N \cap \gamma_i(G) \leqslant N \cap \mathrm{Z}(G)$$

so $N \cap \mathrm{Z}(G) \neq \mathbf{1}$.