Final Exam, May 8, 2018 **Math 21b, Spring 2018**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- All matrices are real matrices unless specified otherwise.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader cannot be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A and B are orthogonal matrices, then so is $A + B$.

Solution:
Take any orthogonal matrix A and define $B = -A$.

2) T F Every real 2×2-matrix has a real eigenvalue.

Solution:
Take a rotation matrix.

3) T F If $\{v_1, v_2\}$ is a basis of R^2 and A is an invertible (2 x 2)-matrix, then $\{Av_1, Av_2\}$ is also a basis of R^2.

Solution:
If S is the matrix of the first basis and T the matrix with columns Av_1, Av_2, then $T = AS$. Now T is invertible because both A and S are invertible.

4) T F A basis for the kernel of the operator $(D-1)^2$ on C^∞ is $\{e^{\lambda t}, te^{\lambda t}\}$ for $\lambda = 1$.

Solution:
This is part of the cookbook procedure. It can also be seen by inverting the operator $D - 1$ twice.

5) T F All symmetric matrices with positive matrix entries have eigenvalues which are all real and positive.

Solution:
A reflection for example has also a negative eigenvalue.

6) T F If A is a reflection matrix, then A is invertible.

Solution:
Its inverse is A.

7) T **F** If A is a symmetric $n \times n$ matrix such that $A^6 = I_n$, then A is the identity matrix I_n.

Solution:
The matrix A could also be reflection for example.

8) **T** F The operator $T = D^2 + 1$ defines a linear transformation $f \to Tf$ from C^∞ to C^∞.

Solution:
Yes, these type of operators were often used in the entire course.

9) **T** F If A and B are $n \times n$ symmetric matrices, then $A^2 - B^2$ is symmetric.

Solution:
A^2 is symmetric and B^2 is symmetric and so the difference.

10) **T** F In the Fourier series expansion of the function $f(x) = x^5$ on $[-\pi, \pi]$ the coefficient a_n of $\cos(nx)$ is zero for every n.

Solution:
The function is an odd function.

11) T **F** If A and B are diagonalizable over the reals, then $A + B$ is diagonalizable over the reals.

Solution:
Start with C, the shear, then take $A = \text{Diag}(1000, 0)$ and $B = C - A$, then both B and A are diagonalizable but $A + B$ is not.

12) T **F** Every 5×4 matrix has a non-zero vector in its kernel.

Solution:
There can be 4 leading ones.

13) T **F** If A is a square matrix such that $\ker(A)$ is contained in $\text{im}(A)$, then $A^2 = 0$.

Solution:
Take $A = I_n$, then the kernel is contained in the image but A^2 is not zero.

14) T **F** The a_0 Fourier coefficient of the function $f(x) = e^{-x^2}$ is zero.

Solution:
That was in the lab. It is not zero

15) **T** F If an invertible square matrix A has two QR decompositions $A = Q_1 R_1$ and $A = Q_2 R_2$, then $R_1 = R_2$.

Solution:
You have proven this in a homework. The QR decomposition is unique for invertible matrices.

16) **T** F The eigenvalue 1 of a reflection at a line in $\mathbb{R}^2$ cannot have geometric multiplicity 2.

Solution:
The matrix is diagonalizable. It would be conjugated to the identity matrix.

17) T **F** If $(d/dt)\vec{x}(t) = A\vec{x}(t)$ is an asymptotically stable continuous dynamical system, then each eigenvalue λ of A satisfies $|\lambda| < 1$.

Solution:
This is a mix-up of discrete and continuous dynamical systems.

18) **T** F The solutions to the heat equation $u_t(x,t) = u_{xx}(x,t)$ with $u(x,0) = \sin(14x)$ converge to 0 as $t \to \infty$.

Solution:
True, from the solutions

19) **T** F If A is a 3×3 matrix and u, v are vectors such that $Av = 5v$ and $Au = -u$ then $A(3v + 2u) = 15v - 2u$.

Solution:
Just be computation.

20) T F There are real 2×2 matrices A and B such that $AB - BA = I_2$.

Solution:
This was the hardest problem. We softened it a bit by mentioning in the global review that $tr(AB) = tr(BA)$ but it was still not obvious to use the idea of looking at the trace of both sides. The left hand side has zero trace, the right hand side has trace 2. This is a very interesting example. It shows that the **Heisenberg anti-commutation** relations can not be realized in finite dimensions.

Problem 2) (10 points) No justifications needed

a) (3 points) We check some properties of the specific matrix A which is given below. Check each box which applies: (By “diagonalizable”, we mean “diagonalizable over the reals”):

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Property	Check if applies
A is a projection	
A is a reflection	
A is orthogonal	
A is invertible	
A is symmetric	
A is diagonalizable	

Solution:
Only the last three (invertible, symmetric and diagonalizable) apply.

b) (3 points) Pick the statements which are true for every non-invertible 2×2 matrix A.

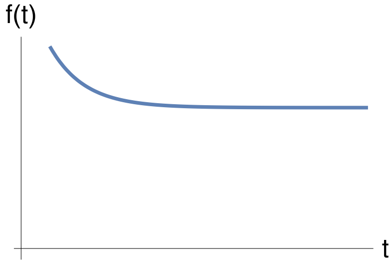
Statement	Check if true
The trace is zero	
The determinant is zero	
The trace of A is an eigenvalue of A	
The determinant of A is an eigenvalue	
The two columns are parallel	
The two rows are parallel	

Solution:
All apply except the first statement about the trace.

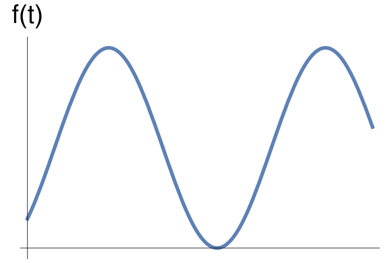
c) (4 points) Match the differential equations with possible solution graphs. Each equation matches exactly one graph.

Enter A-D	Differential equation
	$f''(t) = 1$
	$f'(t) = 1$

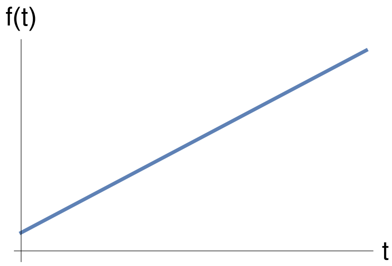
Enter A-D	Differential equation
	$f'(t) = -f(t) + 1$
	$f'''(t) = -f'(t)$



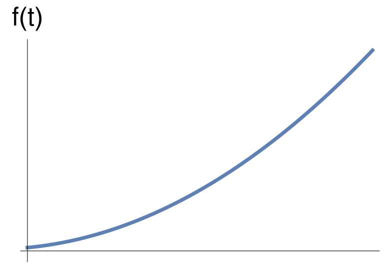
A)



B)



C)



D)

Solution:

Enter A-D	Differential equation
D	$f''(t) = 1$
C	$f'(t) = 1$

Enter A-D	Differential equation
A	$f'(t) = -f(t) + 1$
B	$f'''(t) = -f'(t)$

Problem 3) (10 points) No justifications needed

a) (3 points) Which of the following matrices are diagonalizable over the reals? Check each box which applies.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:
Only the middle box applies. This is a symmetric matrix which is diagonalizable.

b) (2 points) Which of the following two matrices has an eigenvalue which is not real? They are "still lives" in the "game of life". Which one? Enter either "Loaf" or "Boat" in the following box:

$$\text{Loaf} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Boat} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Solution:
it is the Loaf.

c) (2 points) Linear or not linear? In all cases, we deal with functions f in the linear space $X = C^\infty_{\text{per}}([-\pi, \pi])$ for which $a_n(f), b_n(f)$ are the Fourier coefficients of f and $||f||$ is the length of the function as defined in Fourier theory.

Space	linear	nonlinear	Transformation	linear	nonlinear
$\{f \in X \mid f(1) = b_1(f)\}$			$Tf(x) = a_1(f)f$		
$\{f \in X \mid f'(1) = 0\}$			$Tf(x) = \|\|f\|\| \sin(x)$		

Solution:
The left parts are linear, the right parts were non-linear

d) (3 points) A continuous function on $[-\pi, \pi]$ has the Fourier series

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) .$$

Check a box, if the statement above the box is true for every set of functions specified to the left.

Function	Check if $a_0 = 0$	Check if all $a_n = 0, n > 0$	Check if all $b_n = 0$
f is even			
f is odd			
f has no real roots			

Solution:

Function	Check if $a_0 = 0$	Check if all $a_n = 0, n > 0$	Check if all $b_n = 0$
f is even			x
f is odd	x	x	
f has no real roots			

Problem 4) (10 points)

We look for 4 numbers x, y, z, w. We know their sum is 20 and that their "super sum" $x-y+z-w$ is 10. As a matter of fact these two equations form a system $Ax = b$ which defines a 2-dimensional plane V in 4-dimensional space.

a) (6 points) Find the solution space of all these numbers by row reducing its augmented matrix $B = [A|b]$ carefully.

$$B = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 20 \\ 1 & -1 & 1 & -1 & 10 \end{array}\right] .$$

b) (4 points) Find two linearly independent vectors which are perpendicular to the kernel of A.

Solution:
a) Row reduction leads to

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 15 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

The solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

b) To be perpendicular to the kernel of A means being in the image of A^T. So, the rows of A are the solution

$$\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \} .$$

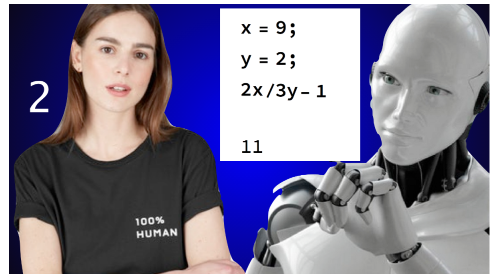
Problem 5) (10 points)

People on social media have been in war about expressions like $\boxed{2x/3y - 1 \text{ if } x = 9 \text{ and } y = 2}$. Computers and humans disagree: most humans get 2, while most machines return 11. A psychologist investigates whether the size of the numbers influences the answer and asks people. (Previous research has shown that machines are not impressed by size). This needs data fitting:

Using the least square method, find those a and b such that

$$\frac{ax}{3y} - b = 2$$

best fits the data points in the following table:



x	y
9	3
6	1
-3	1
0	1

Solution:
Write down the systems of equations (don't leave out any of the equations) $a - b = 22a - b = 2 - a - b = 2 - b = 2$. The matrix is

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}$$

We have $b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$. The solution $\vec{x} = (A^T A)^{-1} A^T b$ is $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$. This is not a surprise as $a = 0, b = -2$ exactly solves the problem.

Problem 6) (10 points)

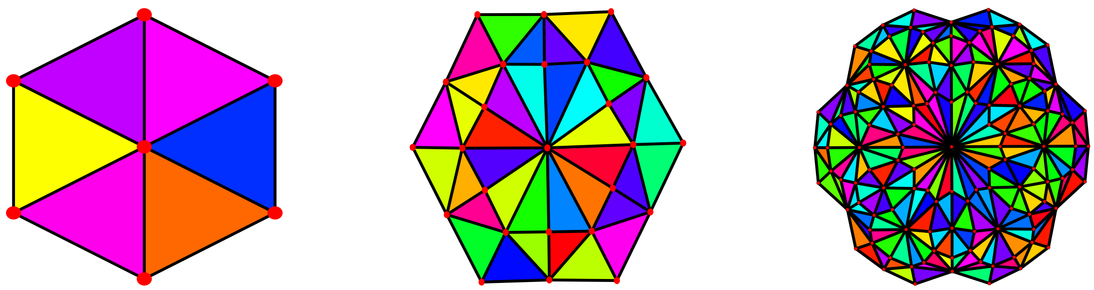
Let $\vec{x} = \begin{bmatrix} v \\ e \\ f \end{bmatrix}$ denote the number of vertices, edges and faces of a polyhedron. During a **Barycentric refinement**, this vector transforms as

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} \vec{x} \, .$$

a) (5 points) Verify that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ are eigenvectors of A and find their eigenvalues.

b) (5 points) Write down a closed form solution of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$

with the initial condition $\begin{bmatrix} v \\ e \\ f \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 6 \end{bmatrix}$.



Solution:
a) Just compute $A\vec{v}_1$ and see that it is a multiple of $\vec{v}_1$. Similarly, do that with $A\vec{v}_2$ and $A\vec{v}_3$. The eigenvalues are $1, 2, 6$. b) Write

$$\begin{bmatrix} 7 \\ 12 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

giving $c_1 = 1$, $c_2 = 3$ and $c_3 = 3$. We can now write down the closed form solution

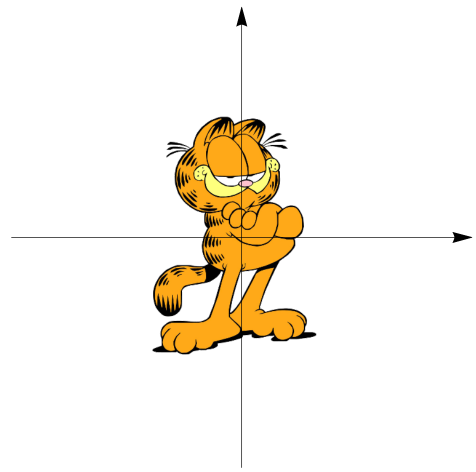
$$1^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2^t \cdot 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 6^t \cdot 3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} .$$

Problem 7) (10 points)

The **Arnold cat map** is $T\vec{v} = A\vec{v}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

It is an icon of chaos theory.



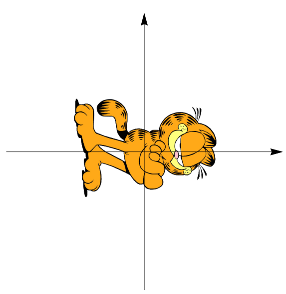
a) (2 points) What is the characteristic polynomial of A?

b) (2 points) Find the eigenvalues and eigenvectors of A. Don't expect integer answers.

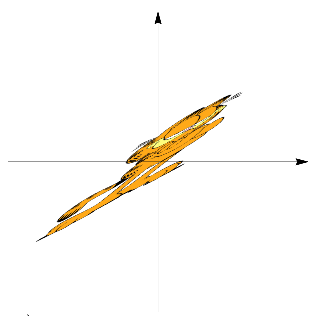
c) (2 points) Is the discrete dynamical system defined by A asymptotically stable or not?

d) (2 points) Write down an orthogonal matrix S and a diagonal matrix B such that $B = S^{-1}AS$.

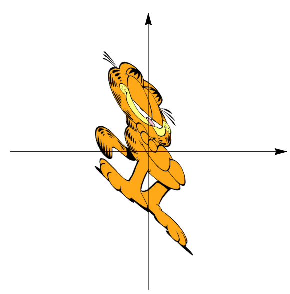
e) (2 points) Garfield G seen to the right has been transformed by the cat map T. Which one of the pictures a),b),c),d) represents the transformed picture $T(G)$?



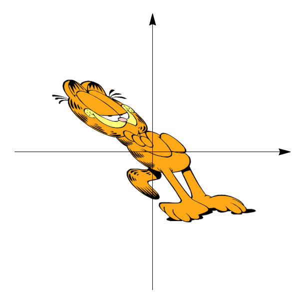
a)



b)



c)



d)

Solution:
a) $\lambda^2 - 3\lambda + 1 = 0$.
b) $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$. The eigenvectors are $\begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$ and $\begin{bmatrix} -1 \\ \frac{\sqrt{5}+1}{2} \end{bmatrix}$.
c) No, the absolute value of the eigenvalues is bigger than 1.
d) The matrix S is the matrix in which normalized eigenvectors are put as columns. The matrix B is the diagonal matrix with eigenvalues in the diagonal.
e) Garfield has been transformed in picture b).

Problem 8) (10 points)

The following configuration is called the "Beacon Oscillator" in the **Game of Life**.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (2 points) What is the rank and the nullity of A?

b) (4 points) Find a basis for the kernel and a basis for the image of A.

c) (4 points) The following matrix is called the "glider configuration" in the **Game of life**.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the inverse of A using row reduction.

Solution:
a) The rank is 4, the nullity is 2.
b) A basis of the kernel is $\mathcal{B} = \{e_1, e_6\}$. A basis for the image are the 4 middle columns of the matrix.
c) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Problem 9) (10 points)

Remember to give computation details. Answers alone can not be given credit.

a) (2 points) The following matrix displays the solution of the Cellular automaton 10. Find its determinant

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

b) (2 points) Find the determinant of

$$B = \begin{bmatrix} 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

c) (2 points) Find the determinant of

$$C = \begin{bmatrix} 1 & 2 & 3 & 8 & 8 \\ 4 & 5 & 0 & 8 & 8 \\ 6 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} .$$

d) (2 points) Find the determinant of

$$D = \begin{bmatrix} 11 & 2 & 3 & 2 & 1 \\ 1 & 12 & 3 & 2 & 1 \\ 1 & 2 & 13 & 2 & 1 \\ 1 & 2 & 3 & 12 & 1 \\ 1 & 2 & 3 & 2 & 11 \end{bmatrix} .$$

e) (2 points) Find the determinant of $E = 2Q + 5Q^{-1} + 7I$: (you can leave it in terms of eigenvalues of the basic circulant matrix Q you have seen. No simplifications are required):

$$E = \begin{bmatrix} 7 & 2 & 0 & 0 & 5 \\ 5 & 7 & 2 & 0 & 0 \\ 0 & 5 & 7 & 2 & 0 \\ 0 & 0 & 5 & 7 & 2 \\ 2 & 0 & 0 & 5 & 7 \end{bmatrix} .$$

Solution:
a) Patterns, 6 upcrossings, det=1
b) Patterns or row reduction $det = -120$.
c) Partitioned. $det = -450$.
d) Build $B = A - 10$ which has eigenvalues $0, 0, 0, 0, 9$, so that B has eigenvalues $10, 10, 10, 10, 19$ which is 190000
e) The determinant is the product of the eigenvalues $\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$ where $\lambda_k = 7 + 2\exp(2\pi i k/5) + 5\exp(-2\pi i k/5)$.

Problem 10) (10 points)

Find the general solution to the following differential equations:

a) (1 point)

$$f'(t) = 1/(t+1)$$

b) (1 point)

$$f''(t) = e^t + t$$

c) (2 points)

$$f''(t) + f(t) = t + 2$$

d) (2 points)

$$f''(t) - 2f'(t) + f(t) = e^t$$

e) (2 points)

$$f''(t) - f(t) = e^t + \sin(t)$$

f) (2 points)

$$f''(t) - f(t) = e^{-3t}$$

Solution:
a) $\log(t+1) + C$
b) $e^t + t^3/6 + C_1 t + C_2$.
c) $C_1 \cos(t) + C_2 \sin(t) + t + 2$
d) $C_1 e^t + C_2 t e^t + t^2 e^t/2$
e) $C_1 e^t + C_2 e^{-t} + te^t/2 - 1/2 \sin(t)$
f) $C_1 e^t + C_2 e^{-t} + e^{-3t}/9$

Problem 11) (10 points)

We consider the nonlinear system of differential equations

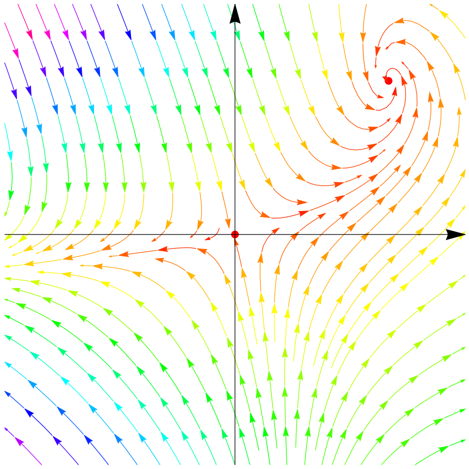
$$\begin{aligned} \frac{d}{dt}x &= x + y - xy \\ \frac{d}{dt}y &= x - 3y + xy \ . \end{aligned}$$

a) (2 points) Find the equilibrium points.

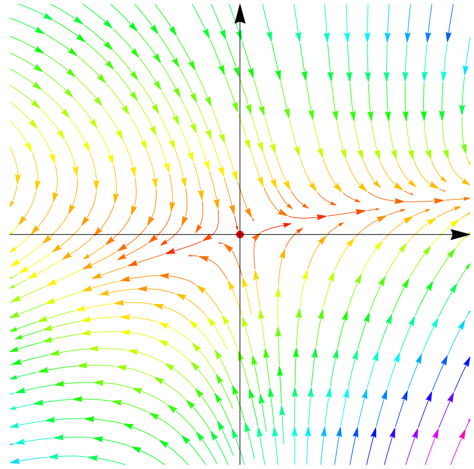
b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Use the Jacobean matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.

d) (2 points) Which of the diagrams A-D is the phase portrait of the system above?



A



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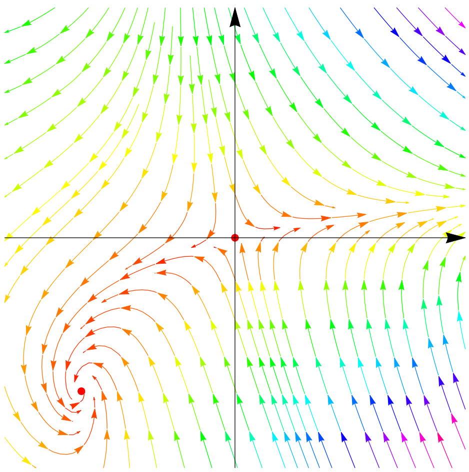
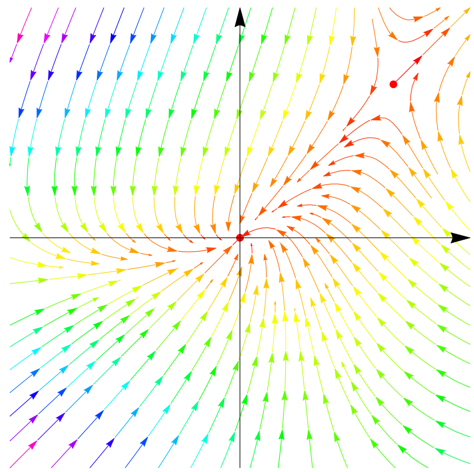


C



D

Solution:
a) The equilibrium points are $(0,0)$ and $(2,2)$.
b) The Jacobian matrices are

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 3 & -1 \end{bmatrix}$$

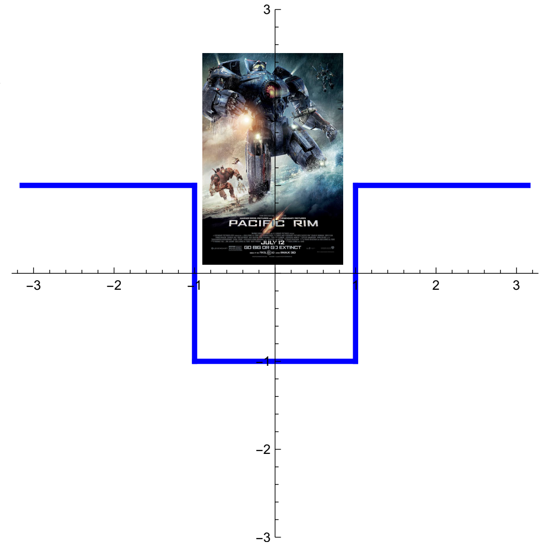
c) The first equilibrium point is unstable, the second is stable. d) We have phase portrait A.

Problem 12) (10 points)

a) (6 points) Find the **Fourier series** of the function which is 1 if $|x| > 1$ and -1 else. We call it the **Pacific rim** function.

$$f(x) = \begin{cases} 1 & |x| > 1 \\ -1 & |x| \leq 1 \end{cases} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



b) (4 points) Find the value of the sum of the squares of all the Fourier coefficients of f.

Solution:
a) The function is even. It has a cos-series. We have

$$a_n = \frac{2}{\pi}\int_0^1 -\cos(nx)\, dx + \frac{2}{\pi}\int_1^\pi \cos(nx)\, dx = -4\sin(n)/(\pi n) .$$

and

$$a_0 = \frac{2}{\pi}\int_0^1 -1/\sqrt{2}\, dx + \frac{2}{\pi}\int_1^\pi 1/\sqrt{2}\, dx$$

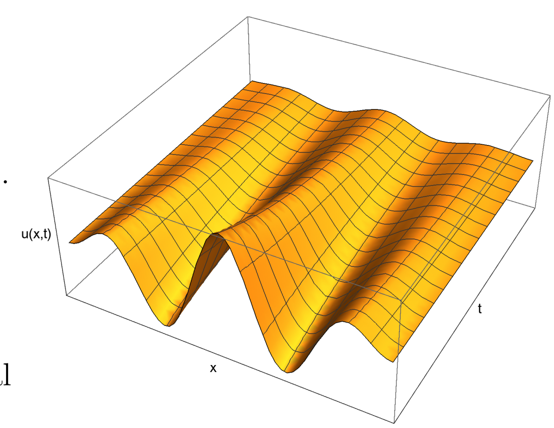
which is $-\sqrt{2}/\pi + (\pi - 1)\sqrt{2}/\pi$.
It could be simplified to $(\pi - 2)\sqrt{2}/\pi$.
b) By Parseval, we know the sum of the squares $\sum_{n=0}^\infty a_n^2$ equal to $(2/\pi)\int_0^\pi 1^2\, dx = \boxed{2}$.

Problem 13) (10 points)

a) (3 points) Solve the standard **heat equation**

$$u_t = 9u_{xx}$$

with initial condition $u(x, 0) = 4\sin(x) + 5\sin(2x)$.



b) (2 points) Use Parseval's equality to find $||u(x, 1)||^2$.

c) (5 points) Solve the **modified wave equation**

$$u_{tt} = 9u_{xx} - 2u + 1$$

with initial condition $u(x, 0) = 7\sin(5x)$ and the initial velocity

$$u_t(x, 0) = x .$$

Solution:
a) The eigenvalue of $9D^2$ is $-9n^2$ with eigenfunction $\sin(nx)$. The function is already be given as a Fourer series. The solution is

$$4\sin(x)e^{-9t} + 5\sin(2x)e^{-36t}$$

b) Parseval tells that the answer is $(4e^{-t})^2 + (5e^{-37t})^2$.
c) There are three components of the solution. The solution u_{pos} with initial position the solution u_{vel} for the initial velocity and then u_{par} the particular solution. The position solution is

$$u_{pos}(x,t) = 7\sin(5x)\cos(15t)$$

The velocity solution is

$$u_{vel}(x,t) = \sum_n \frac{2(-1)^{n+1}}{n} \frac{\sin(3nt)}{n} \sin(nx) \ .$$

The particular solution is the solution to

$$u_{tt} = 2u + 1$$

which is by cookbook

$$u_{part}(x,t) = C_1\cos(\sqrt{2}t) + C_2\sin(\sqrt{2}t)/\sqrt{2} = \cos(\sqrt{2}t)/2 - 1/2 \ .$$

as the initial condition $u_{part}(x,0) = 0, u'_{part}(x,0) = 0$ determines the constants C_1, C_2. The solution is

$$u(x,t) = 7\sin(5x)\cos(15t) + \sum_{n=1} \frac{2(-1)^{n+1}}{n} \frac{\sin(3nt)}{n} \sin(nx) + \frac{\cos(\sqrt{2}t) - 1}{2} \ .$$

Solution:
Remark: In practice exams, reviews and homework we had only seen examples where the operator on the right hand side kills every function of time only. Only a handful of students could solve the problem completely as above. Evenso it does not quite solve the problem, we also gave full credit for a solution in which the operator $9D^2 - 2$ was taken and the inhomogeneous part is solved by $u_{tt} = 1$. The function (which is not quite a solution however) is then

$$u(x,t) = 7\sin(5x)\cos(\sqrt{223}t) + \sum_{n=1} \frac{2(-1)^{n+1}}{n} \frac{\sin(\sqrt{9n^2-2}t)}{\sqrt{9n^2-2}} \sin(nx) + \frac{t^2}{2} \ .$$