MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) **T** F — If A is a real $n \times n$ matrix, then $A + A^T$ is diagonalizable.

Solution:
The matrix $A + A^T$ is symmetric and therefore diagonalizable.

2) **T** F — The equation $z^3 = -1$ has three different complex solutions z and $z = -1$ is one of them.

Solution:
The solutions are of the form $\exp(i(\pi + 2\pi k/3))$, where $k = 0, 1, 2$.

3) T **F** — There is a 2×2 projection matrix A that projects onto a line and a 2×2 reflection matrix B that rotates about a point, so that AB is a rotation matrix.

Solution:
The resulting matrix AB is not invertible and can not be a rotation matrix.

4) T **F** — The transformation $T(f)(x) = f(f(x))$ is linear on the space C^∞ of all smooth functions

Solution:
Check the three conditions. While $T(0) = 0$, it is not true that $T(2f) = 2f$ for $f(x) = x^2$ for example.

5) **T** F — If a continuous function f defined on the interval $[-\pi, \pi]$ is both even and odd, then it must be constant function.

Solution:
Fourier theory shows this. The function is perpendicular to all even functions and all odd functions.

6) T **F** — If A is a 2×2 matrix, then the characteristic polynomial of AA^T is $\lambda^2 + c\lambda$ for some real constant c.

Solution:
False, it can be invertible.

7) T F The function $f(t) = e^t$ is an eigenfunction with eigenvalue 1 of the linear operator $T = D^2$ where $Df = f'$.

Solution:
It is indeed an eigenfunction to the eigenvalue 1.

8) T F The initial value problem $f''(x) + f(x) = \sin(x), f''(0) = 1, f'(0) = 1$ has exactly one solution.

Solution:
We can use the differential equation to compute $f(0)$ and so have exactly one solution.

9) T F The transformation $L(A) = A + A^T$ is a linear transformation on the space of $n \times n$ matrices M_n and L has only the eigenvalues 0 and 2.

Solution:
We check three conditions for linearity and note that symmetric matrices are eigenvectors to the eigenvalue 2 and antisymmetric matrices are eigenvectors to the eigenvalue 0.

10) T F The set X of smooth functions $f(x, y, z)$ which satisfy the $f_{xx} + f_{yy} + f_{zz} = f$ is a linear space.

Solution:
Yes, the sum also satisfies this differential equation. Also a scaled function satisfies the differential equation. And the zero function also satisfies this differential equation.

11) T F If A is a 5×5 matrix that has rank 2, then the eigenvalue 0 has algebraic multiplicity 3.

Solution:
The kernel is 3 dimensional by the rank-nullity theorem. But the algebraic multiplicity can be larger.

12) T F Every equilibrium point of a nonlinear system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is located on at least one nullcline.

Solution:
Equilibrium points are on the intersection of nullclines.

13) T **F** The transformation $T(A) = \text{rank}(A)$ is linear from the space M_2 of all real 2×2 matrices to R.

Solution:
Already $T(2I) = 2T(I)$ is not true.

14) **T** F The QR decomposition of a $n \times n$ reflection matrix A is $A = QR$, where $Q = A$ and $R = I_n$.

Solution:
It is already the QR decomposition.

15) T **F** If all eigenvalues of a 2×2 matrix A are zero then A is the zero matrix.

Solution:
The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a counter example.

16) T **F** The discrete dynamical system $x(t+1) = 7x(t) + 3\,x(t-1)$ has the property that $|x(t)|$ stays bounded for all initial conditions $(x(0), x(1))$.

Solution:
The trace of the system is 7. The sum of the two eigenvalues being 7 makes it impossible that both eigenvalues satisfy $|\lambda| < 1$.

17) **T** F $||6\sin(x) + 8\cos(9x)|| = 10$, where $||f|| = \sqrt{\langle f, f\rangle}$ is the length of the function f with respect to the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

Solution:
This is a direct consequence of the Perseval identity. There are only 2 nonzero Fourier coefficients b_1 and a_9.

18) **T** F Any reflection has real eigenvalues and an orthonormal eigenbasis.

Solution:
A reflection has eigenvalues 1 and -1. It is given by a symmetric matrix.

19) T F Given a $n \times m$ matrix A with trivial kernel. If $A(A^TA)^{-1}A^Tb = b$ for all b, then $n = m$.

Solution:
The condition means that the projection $P = A(A^TA)^{-1}A^T$ is invertible, which implies that P is the identity matrix.

20) T F If A, B are 2×2 matrices and a system $Ax = 0$ has infinitely many solutions and the system $Bx = 0$ has exactly one solution, then $ABx = 0$ has infinitely many solutions.

Solution:
The first assumption means that A has a nontrivial kernel. The second condition means that B is invertible. Whenever x has the property that Bx is in the kernel of A, then it solves the equation $ABx = 0$.

Problem 2) (10 points) no justifications needed

a) (5 points) Circle all the matrices which are similar to the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

b) (5 points) Match the differential equations with solution functions. Enter A-E into the boxes to the left. Every choice A-E appears exactly once (despite the fact that two entries on the left appear in pairs):

	$f''(t) + f(t) = t$
	$f''(t) - f(t) = t$
	$f''(t) = t$
	$f''(t) - f(t) = t$
	$f''(t) + f(t) = t$

A) $f(t) = e^t - t$.

B) $f(t) = t + \cos(t) - \sin(t)$

C) $f(t) = 1 - t + t^3/6$

D) $f(t) = e^t - e^{-t} - t$

E) $f(t) = t$

Solution:
a) All matrices which have eigenvalues $1, -1$ are similar to the given matrix. These are the matrices A, B.
b) B A C D E. It is possible to switch the first with the last or the second with the second last.

Problem 3) (10 points) no justifications needed

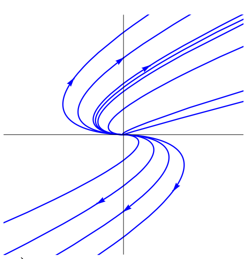
a) (5 points) λ_1, λ_2 λ_3 are the eigenvalues of a 3×3 matrix A which represents a transformation T in space. Which transformation belongs to these eigenvalues? All except one of the transformations a)-f) appear. Each of the five appears exactly once.

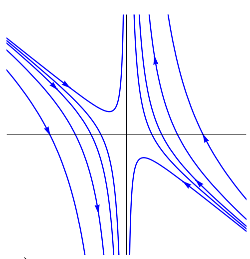
λ_1	λ_2	λ_3	Enter a)-f) here
1	1	1	
-1	-1	1	
0	1	0	
0	1	1	
1	-1	1	

a) projection onto a plane
b) reflection at a plane
c) reflection at a line
d) projection onto a line
e) identity matrix
f) rotation about an axis

b) (5 points) Enter 1)-6) in the first columns and check the boxes. "asymptotically stable" is abbreviated by "stable" and "diagonalizable" means diagonalizable over the complex numbers. The phase portraits belong to the continuous system $x' = Ax$. All except one of the phase portrait can be matched. Each of the five appears exactly once.

matrix	phase 1)-6)	$\frac{d}{dt}x = Ax$ stable	$x(t+1) = Ax(t)$ stable	A diagonalizable
$\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$				
$\begin{bmatrix} -2 & 0 \\ 4 & 3 \end{bmatrix}$				
$\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$				
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$				
$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$				


1)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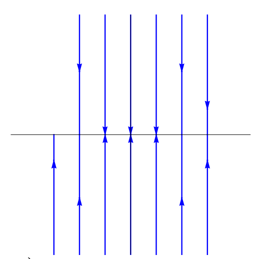
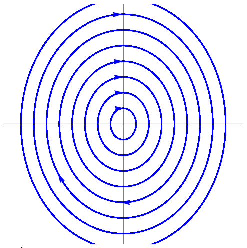
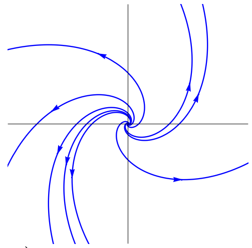
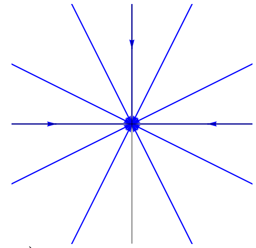

3)



4)



5)



6)

Solution:
a) The solutions are e),c),d),a),b)
b)

matrix	phase 1)-6)	$\frac{d}{dt}x = Ax$ stable	$x(t+1) = Ax(t)$ stable	A diagonalizable
$\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$	1			x
$\begin{bmatrix} -2 & 0 \\ 4 & 3 \end{bmatrix}$	2			x
$\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$	3			x
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	5			x
$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	6	x		x

Problem 4) (10 points)

Find all the solutions of the system of linear equations for the 5 variables x, y, z, u, v using row reduction.

$$\left| \begin{array}{ccccccccccc} x & & & + & z & - & u & + & v & = 2 \\ x & + & y & + & z & + & u & + & v & = 6 \\ x & + & y & - & z & - & u & + & v & = 0 \end{array} \right|$$

Solution:
The augmented matrix is

$$B = \left| \begin{array}{ccccccc} 1 & 0 & 1 & -1 & 1 & | & 2 \\ 1 & 1 & 1 & 1 & 1 & | & 6 \\ 1 & 1 & -1 & -1 & 1 & | & 0 \end{array} \right| .$$

Row reduction produces

$$\mathrm{rref}(B) = \left| \begin{array}{ccccccc} 1 & 0 & 0 & -2 & 1 & | & -1 \\ 0 & 1 & 0 & 2 & 0 & | & 4 \\ 0 & 0 & 1 & 1 & 0 & | & 3 \end{array} \right| .$$

There are first leading 1 in this matrix and two free variables. Lets call them r, s. The general solution can be written as $x = -1 + 2r - s, y = 4 - 2r, z = 3 - r, u = r, v = s$. In vector form, this is

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

Problem 5) (10 points)

a) (7 points) Find the best function $ax^2 + b2^x = y$ which fits the data points $(1, 2), (0, 1), (2, 4)$ using the least square method.

b) (3 points) With the data points $(0, 1), (0, 2), (0, 0)$, there is more than one minimal solution. Examples are $x^2 + 2^x = y$ and $2x^2 + 2^x = y$. Why does the least square method fail in this case? Choose one of the following explanations. No further explanations are needed in this part b).

The matrix A in the least square solution formula is undefined.	
The matrix A is not invertible.	
The matrix $A^T A$ is not invertible.	

Solution:
a) We write down a system of linear equations which pretends that all points are on the curve

$$\begin{aligned} a \cdot 1^2 + b2^1 &= 2 \\ a \cdot 0^2 + b2^0 &= 1 \\ a \cdot 2^2 + b2^2 &= 4 \,. \end{aligned}$$

The corresponding matrix A and vector b is

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} .$$

To crank this into the formula for the least square solution

$$\begin{bmatrix} a \\ b \end{bmatrix} = x = (A^T A)^{-1} A^T b \,,$$

compute $A^T A = \begin{bmatrix} 17 & 18 \\ 18 & 21 \end{bmatrix}$ and $(A^T A)^{-1} = \begin{bmatrix} 21 & -18 \\ -18 & 17 \end{bmatrix}/33$ and $A^T b = \begin{bmatrix} 18 \\ 21 \end{bmatrix}$ so that $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The best fit is the function $\boxed{y = 2^x}$. It is actually an exact solution. (We did not know when designing the problem, but one or two students noticed).
b) The least square solution fails because A and so $A^T A$ has a kernel. In that case, we can not invert $A^T A$. The answer b) can not be true: already in a), the matrix A is not invertible because it is not an $n \times n$ matrix. For all least square problems, the matrix A is defined. So the correct answer is:

The matrix A in the least square solution formula is undefined.	
The matrix A is not invertible.	
The matrix $A^T A$ is not invertible.	X

Problem 6) (10 points)

a) (6 points) Find all the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

b) (4 points) Find all the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} .$$

Solution:
a) The matrix has the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_4) = \lambda^4 - 1$ which has the solutions $1, i, -1, -i$. Like for any circular matrix, the eigenvector to the eigenvalue λ is

$$\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix} .$$

In this case we have

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} .$$

b) This is a partitioned matrix. The first 4×4 matrix is the one from a). The other two entries are 1×1 matrices and lead to additional eigenvalues 5 and 6. The eigenvalues are $1, i, -1, -i, 5, 6$. The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \\ 0 \\ 0 \end{bmatrix} v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

Problem 7) (10 points)

a) (8 points) Find a closed form solution of the difference equation

$$x_{n+1} = 2y_n - 3x_n$$

$$y_{n+1} = x_n - 2y_n$$

with initial condition $x_0 = 5, y_0 = 3$.

b) (2 points) A system is called **Lyapunov stable** if for all (x_0, y_0) the sequence (x_n, y_n) stays in a bounded region $R(x_0, y_0)$. (This means that no trajectory can run off to infinity.) Check the following boxes. No further reasoning is needed in this part b).

The system is Lyapunov stable	
The system is asymptotically stable	

Solution:
a) The eigenvalues of $A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$ are $\lambda_1 = -4, \lambda_2 = -1$ with eigenvectors

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

We have to write initial condition as a combination of eigenvectors:

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Which gives $c_1 = -2/3, c_2 = 11/3$. We have the general solution

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \frac{-2}{3}(-4)^n \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{11}{3}(-1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

b) The system is neither Lyapunov stable, nor asymptotically stable. An example of a system which is Lyapunov stable is

$$x_{n+1} = 2x_n - 3y_n$$

$$y_{n+1} = x_n - 2y_n .$$

Problem 8) (10 points)

Compute the eigenvalues and an orthonormal eigenbasis for the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} .$$

Solution:
$A - I_3$ has a two dimensional kernel spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

These are the eigenvectors to the eigenvalue 1. There is an eigenvalue 3 with eigenvector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To make the eigenbasis

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

orthonormal, do Gram-Schmidt orthonormalization:

$$w_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} w_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Since the problem unfortunately stated "orthonormal basis", instead of orthonormal eigenbasis, (on Vulcan this would have been implicitly understood but not here on earth), full credit was also given for a correct Gram-Schmidt factorization of the columns of A (which is harder) or reasoning why the standard basis provides an orthonormal basis for the image (only 2 students).

Problem 9) (10 points)

a) (7 points) Find the determinant of the following matrix

$$\begin{bmatrix} 9 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 9 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 9 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 9 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 1 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

Make sure to mention all tools you need to find the answer.
b) (3 points) Find the determinant of a 5×5 matrix X which satisfies

$$AXA = B ,$$

where A is a reflection at a one-dimensional line in R^5 and B is a reflection at a two-dimensional plane in R^5. As usual, justify your answer.

Solution:
a) This is a partitioned matrix and consists of three submatrices The first 5×5 submatrix has eigenvalues $13, 8, 8, 8, 8$ and so determinant $13 \cdot 8^4$. The second submatrix has only one pattern with 5 upcrossings and determinant -120. The third matrix can be row reduced the identity matrix by subtracting rows alone which does not change the determinant. The determinant of the third submatrix is 1. All together, we have the product of the 3 determinants $\boxed{13 \cdot 8^4 \cdot (-120) \cdot 1 = -6389760}$.
b) The matrix A has determinant 1 as can be seen by using a basis which makes the transformation diagonal. Similarly, matrix B has determinant -1. By the product formula for determinants, we have

$$\det(A)\det(X)\det(A) = \det(B) .$$

The determinant of X is $\boxed{-1}$. An other line of reasoning would be that a reflection is its own inverse so that X is similar to B and has the same determinant as B.

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (3 points)

$$f'' + 16f = \cos(4x), f(0) = f'(0) = 1$$

b) (2 points)

$$f'' = \sin(4x), f(0) = f'(0) = 1$$

c) (3 points)

$$f'' - f' + f = 1, f(0) = 2, f'(0) = 3$$

d) (2 points)

$$f'' - 2f' + f = 0$$

Solution:
a) This is a driven harmonic oscillator. The inhomogeneous driving force is an eigen mode=eigenfunction, which leads to resonance. The homogeneous solution is $C_1\cos(4t) + C_2\sin(4t)$ a special solution is found by plugging in $At\sin(4t) + Bt\cos(4t)$ and finding the constant $A = 1/8, B = 0$. The general solution without initial conditions is

$$C_1\cos(4t) + C_2\sin(4t) + t\sin(4t)/8$$

Plugging in the initial conditions fixes C_1, C_2 and leads to $\boxed{f(t) = \cos(4t) + t\sin(4t)/8 + \sin(4t)/4}$.
b) This is a situation, where we can best use the operator method which means integrating twice. We get $C_1 + C_2t - \sin(4t)/16$ as the general solution without initial condition. Plugging in the initial conditions fixes C_1, C_2 and leads to $\boxed{f(t) = 1 + 5t/4 - \sin(4t)/16}$.
c) To get the homogeneous solution, we factor $D^2 - D + 1 = (D - (1/2 + i\sqrt{3}/2))(D - (1/2 - i\sqrt{3}/2))$ leading to $C_1e^{t/2}\cos(\sqrt{3}/2) + C_2e^{t/2}\sin(\sqrt{3}/2)$. For a particular solution, we try a constant A and get $A = 1$. The general solution without initial condition is

$$1 + C_1e^{t/2}\cos(\sqrt{3}t/2) + C_2e^{t/2}\sin(\sqrt{3}/2)\ .$$

Fixing the constants with the initial condition leads to $\boxed{1 + e^{t/2}\cos(\sqrt{3}t/2) + (5/\sqrt{3})e^{t/2}\sin(\sqrt{3}t/2)}$.
d) The factorization $D^2 - 2D + 1 = (D - 1)^2$ gives the homogeneous solution

$$C_1e^t + C_2te^t\ .$$

There is no inhomogenity, and no initial condition so that this is the general solution.

Problem 11) (10 points)

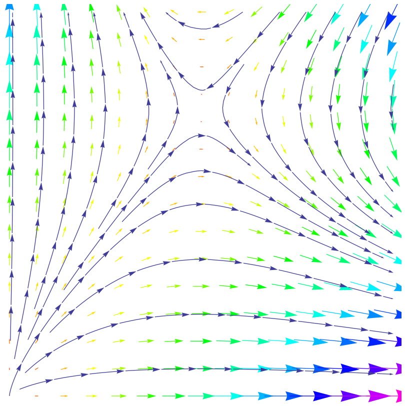
The following nonlinear dynamical system appears as a catalytic network in biology

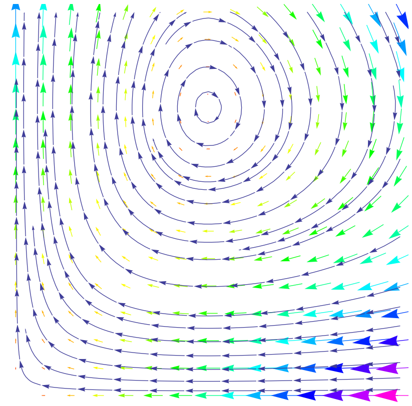
$$\begin{aligned} \frac{d}{dt}x &= 2x - xy \\ \frac{d}{dt}y &= 3y - xy \end{aligned}$$

a) (3 points) Find the equations of the null clines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

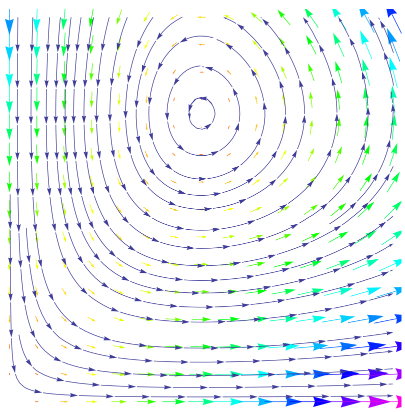
c) (3 points) Which of the phase portraits A,B,C,D below belongs to the above system?

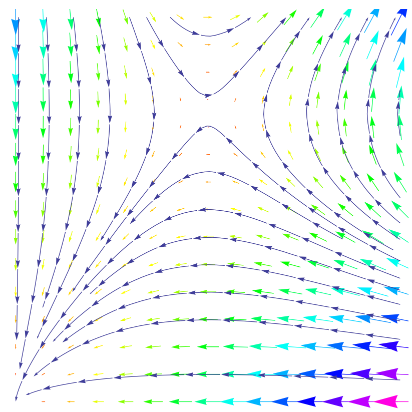




A

B





C

D

Solution:
a) The x-nullcline are $2x - xy = 0$ which is the union of lines $x = 0$ and $y = 2$, the y-nullclines are $3y - xy = 0$ which is the union of lines $y = 0$ and $x = 3$. The intersection of different nullclines gives the 2 equilibrium points $(0,0), (3,2)$.
b) The Jacobian matrix is

$$J(x,y) = \begin{bmatrix} 2-y & -x \\ -y & 3-x \end{bmatrix} .$$

At the four points, we get

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, J(3,2) = \begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix} .$$

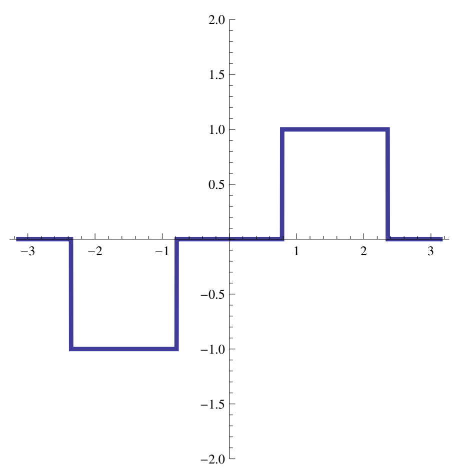
The eigenvalues at $(0,0)$ are $2,3$ which are both positive. This is a "source". All trajectories move away from the origin. The eigenvalues at $(3,2)$ are $-\sqrt{6}, +\sqrt{6}$. This is a "saddle" with one unstable and one stable direction. The system is unstable. In summary, none of the equilibrium points are asymptotically stable.
c) Portrait A) matches the situation. At the equilibrium point $(0,0)$ it has a source and at the equilibrium point $(3,2)$ it has a hyperbolic behavior. Portraits B) and C) are excluded because they have imaginary eigenvalues at the equilibrium point $(3,2)$ and D) is excluded because it has a "sink" at the origin. All trajectories move towards the origin. $\boxed{\text{Answer: A}}$.

Problem 12) (10 points)

a) (7 points) Find the Fourier series of the piecewise constant function

$$f(x) = \begin{cases} 1 & \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} , \\ -1 & \frac{-3\pi}{4} \leq x \leq \frac{-\pi}{4} . \end{cases}$$

The graph of the function is visible to the right.



b) (3 points) Use Parsevals theorem to find the value of the sum

$$\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots .$$

Hint. The Fourier coefficients you found in a) are all zero for even n.

Solution:
a) The function is odd, so that it has a sin-series. The Fourier coefficients are

$$b_n = \frac{2}{\pi}\int_{\pi/4}^{3\pi/4} \sin(nx)\, dx = \frac{2}{\pi}\frac{-\cos(nx)}{n}\Big|_{\pi/4}^{3\pi/4}$$

This is $\pm\frac{2\sqrt{2}}{\pi n}$ for odd n and 0 for even n. The expression above is fine. For b) we don't need the sign.
b) Parseval's identity tells

$$\sum_{n\ odd} \frac{8}{n^2\pi^2} = \frac{2}{\pi}\int_{\pi/4}^{3\pi/4} 1^2\, dx = 1 \ .$$

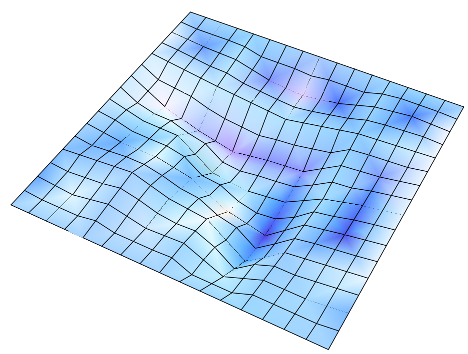
Therefore, $\sum_{n\ odd} \frac{1}{n^2} = \frac{\pi^2}{8}$.

Problem 13) (10 points)

The partial differential equation

$$u_{tt} = T(u) = u_{xx} + u_{yy} - u_{xxxx}$$

is a modification of a two dimensional wave equation and describes a moving membrane $u(x, y, t)$ with $-\pi \le x \le \pi, -\pi \le y \le \pi$.



a) (2 points) Verify that $\sin(nx)\sin(my)$ is an eigenvector of the operator T. Find the corresponding eigenvalue λ_{nm}.

b) (4 points) What is the solution of the above wave type equation if the double sum

$$u(x, y, 0) = \sum_{n,m=1}^{\infty} \frac{1}{nm}\sin(nx)\sin(my)$$

is the initial position of the membrane and the initial velocity of the membrane is zero?

c) (4 points) What is the solution of the above wave type equation if $u_t(x, y, 0) = \sin(4x)\sin(6y)$ is the initial velocity of the membrane and the initial position $u(x, y, 0)$ of the membrane is zero everywhere.

Solution:
a) $T\sin(nx)\sin(my) = (-n^2 - m^2 - n^4)\sin(nx)\sin(my)$ so that $\lambda = (-n^2 - m^2 - n^4)$ is an eigenvalue.
b) If the initial position is an eigenfunction $u = \sin(nx)\sin(my)$, then it follows the ordinary differential equation

$$u_{tt} = \lambda u = -c^2 u$$

with $c = \sqrt{n^2 + m^2 + n^4}$. This "eigenmode"=eigenfunction satisfies the harmonic oscillator equation which has the solution

$$u(x,t) = \cos(ct)u(x,0) = \cos(ct)\sin(nx)\sin(my) = \cos(\sqrt{n^2 + m^2 + n^4}t)\sin(nx)\sin(my) .$$

If the initial position is a linear combination of eigenmodes, then $u(x,t)$ is a sum too

$$u(x,t) = \sum_{n,m} \frac{1}{nm}\cos(\sqrt{(n^2 + mn^2 + n^4)t})\sin(nx)\sin(my) .$$

c) If the initial velocity is an eigenfunction $\sin(nx)\sin(my)$, then the solution to

$$u_{tt} = -c^2 u$$

is

$$u(x,t) = \frac{\sin(ct)}{c}u(x,0) = \frac{\sin(ct)}{c}\sin(nx)\sin(my) = \frac{\sin(\sqrt{n^2 + m^2 + n^4}\ t)}{\sqrt{n^2 + m^2 + n^4}}\sin(nx)\sin(my)$$

Especially, for $n = 4, m = 6$:

$$u(x,t) = \frac{\sin(\sqrt{4^2 + 6^2 + 4^4}\ t)}{\sqrt{4^2 + 6^2 + 4^4}}\sin(4x)\sin(6y) .$$

The final answer is $\boxed{u(x,t) = \sin(\sqrt{308}t)\sin(4x)\sin(6y)/\sqrt{308}}$.

Problem 14) (10 points)

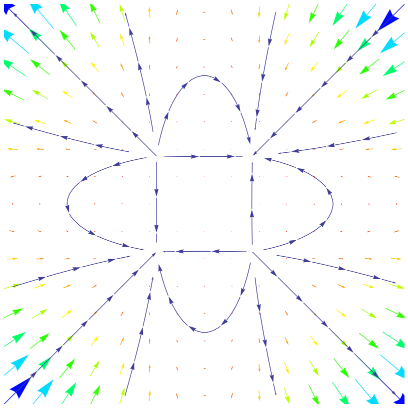
We analyze the following nonlinear system

$$\begin{aligned} \frac{d}{dt}x &= y(1 - x^2) \\ \frac{d}{dt}y &= x(1 - y^2) \end{aligned}$$

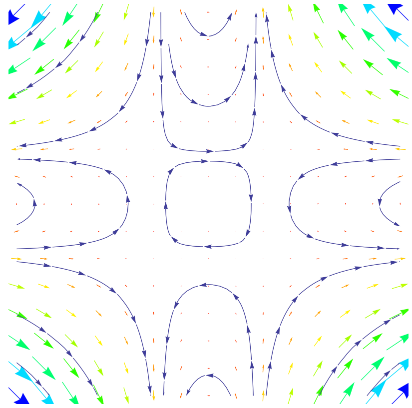
a) (3 points) Find the equations of the null clines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

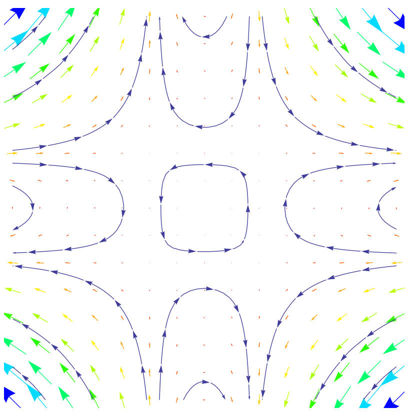
c) (3 points) Which of the phase portraits A,B,C,D below belongs to the above system?



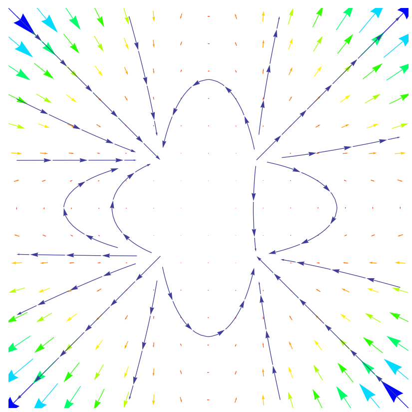
A



B



C



D

Solution:
a) The x-nullcline are $y = 0$ or $x = 1, x = -1$. The y-nullclines are $x = 0$ or $y = 1, y = -1$. This gives 5 equilibrium points $(0,0), (1,1), (1,-1), (-1,1), (-1,-1)$. b) The Jacobian matrix is

$$J(x,y) = \left[\begin{array}{cc} -2xy & 1-x^2 \\ 1-y^2 & -2xy \end{array} \right] .$$

At the equilibrium points, we get

$$J(0,0) = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], J(-1,-1) = J(1,1) = \left[\begin{array}{cc} -2 & 0 \\ 0 & -2 \end{array} \right], J(1,-1) = J(-1,1) = \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array} \right]$$

a) The eigenvalues are $1, -1$ in the first case, $-2, -2$ in the next two cases and $2, 2$ in the last two cases. Only the points $(-1,-1), (1,1)$ are stable.
c) Portrait A) is the only one which has the right stability at the critical points.