Wesleyan University
Department of Mathematics

PhD Qualifying Exam, Written Part: Topology
Profs. Hager and Hovey

August 1, 2005

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts. You must pass each part to pass the exam. "Pass" means "show us you know enough to start trying to write a thesis".

Part One

(1) What are the cardinalities of these sets? $\mathcal{P}(\mathbb{N})$, $\{0,1\}^{\aleph_0}$, $[0,1]$, $(0,1)$, $\{f | f\colon \mathbb{N} \to [0,1] \text{ is a function}\}$. (Here $\mathbb{N}$ is the natural numbers and $\mathcal{P}$ denotes power set).

(2) Prove or disprove.
   (a) If X is second countable and $Y \subseteq X$, then Y is second countable.
   (b) If $f\colon X \to Y$ is continuous and onto, and X is second countable, then Y is second countable.
   (c) The product of any set of second countable spaces is second countable.

(3) Why?
   (a) Why is each of the following not metrizable? The Sorgenfrey line. The one-point compactification of an uncountable discrete space. The product of uncountably many copies of $\mathbb{N}$ (the countable discrete space).
   (b) Why is a compact Hausdorff space completely regular? Why is a completely regular Hausdorff space homeomorphic to a subspace of a compact Hausdorff space?

(4) Give Hausdorff examples of the following:
   (a) X, Y normal, $X \times Y$ not normal.
   (b) X normal, $Y \subseteq X$, Y not normal.
   (c) Infinite compact X with no isolated points. The same with a dense set of isolated points instead.
   (d) Infinite compact X with a point with no countable local base.
   (e) X first countable but not second countable.
   (f) X separable but not metrizable.

(5) Consider the Čech-Stone compactification βX of completely regular Hausdorff X.
   (a) State a set of characteristic properties of βX. (That is, "what is it?" not "how do we make it?")
   (b) What are the cardinalities of $\beta\mathbb{N}$? $\beta S(\omega_1)$? $\beta[0,1]$?

(over)

Part Two

(6) Given a pointed space (X, x_0), define the fundamental group $\pi_1(X, x_0)$ and give a sketch of the proof that it is in fact a group.

(7) Suppose we take an 8-sided polygon and identify the edges according to the scheme $ab^{-1}cdc^{-1}bd^{-1}a$. That is, starting at one vertex and going around the edges clockwise, these are the labels you see, with b^{-1} meaning that b is backwards. Find the fundamental group and the first homology group of this surface and identify it as one of the standard surfaces.

→ (8) Let $\mathcal{U}$ be an open cover of the connected space X, and let $a, b \in X$. Show that there is a some sequence $U_1, \ldots, U_n$ of open sets in $\mathcal{U}$ such that $a \in U_1$, $b \in U_n$, and $U_i \cap U_{i+1}$ is nonempty for $i = 1, 2, \ldots, n-1$. (Hint: Let C be the set of all points b that can be connected to a by such a sequence. What can you say about C?)

(9) Give a space X, define the **suspension** of X, ΣX to be the space obtained by suspending X from two points. Technically, ΣX is the space obtained from the cylinder $X \times I$ by identifying $(x, 1)$ with $(x', 1)$ for all $x, x' \in X$ and $(x, 0)$ with $(x', 0)$. Calculate, with proof, the fundamental group of $\pi_1(\Sigma X)$ assuming X is path-connected.

(10) One of the most important results about topological groups is that the universal cover of a (well-behaved) topological group is another topological group. In this problem, you will prove part of this result.

(a) Suppose we have a diagram

$$\begin{array}{ccc} & & (E, e_0) \\ & & \downarrow p \\ (X, x_0) & \xrightarrow[f]{} & (B, b_0) \end{array}$$

where p is a covering map and all spaces are path connected and locally path connected. Give necessary and sufficient conditions for there to exist a map $\tilde{f}\colon (X, x_0) \to (E, e_0)$ such that $p\tilde{f} = f$. Assuming $\tilde{f}$ does exist, how many choices are there for it?

(b) Now assume G is a path connected, locally path connected, topological group with identity e, multiplication $\mu\colon G \times G \to G$, and a universal cover $p\colon (H, e') \to (G, e)$. Show that there is a map $\mu'\colon H \times H \to H$ such that $p \circ \mu' = \mu \circ (p \times p)$.

(c) Show that the multiplication μ' defined above is associative. To complete the proof that H is a topological group, you would have to prove that H has an identity and a continuous inverse function, but we leave that as homework for the interested student.

Wesleyan University

Department of Mathematics

PhD Qualifying Exam, Written Part: Topology
Profs. Comfort and Hovey

August 30, 2004

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts; answer **three questions from each part** and then as many additional questions as time and ability permit. Partial credit may be awarded, but sparingly; therefore it is in your best interest to answer a number of questions fully, omitting some, rather than to answer all questions incompletely. Throughout this examination, the word "space" refers to a completely regular, Hausdorff space (i.e., a Tychonoff space).

Part One

(1) Prove the following well-known theorem from calculus. If

$$f\colon [a,b] \to \mathbb{R}$$

is continuous, where $a < b \in \mathbb{R}$, then the image of f is a closed bounded interval.

(2) If X_0 is the path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \to X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

(3) Compute the fundamental group, with proof, of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

(4) (a) Give the definition of a covering space. (b) Prove that, if $p\colon (E, e_0) \to (X, x_0)$ is a covering space, and $f\colon (Y, y_0) \to (X, x_0)$ is a continuous map where Y is path-connected, then there exists **at most one** lift $\widetilde{f}\colon (Y, y_0) \to (E, e_0)$ of f with $p\widetilde{f} = f$.

(5) (a) List all the compact connected surfaces, up to homeomorphism. (b) Consider the surface obtained from a hexagon (= 6-sided figure) by identifying the edges in pairs, so that in going clockwise around the hexagon, we see the labels $abaca^{-1}c^{-1}$, where the inverse means the edge is traversed backwards. Which compact connected surface from your list is this, and how do you know?

Part Two

(6) (a) Define carefully any three (standard, well-known) cardinal invariants c which are associated with every topological space. [Note. Do not include on your list the simplest cardinal invariant: $c(X) = |X| =$ the cardinality of X.] (b) For each of the invariants c you defined in (a), prove or disprove this statement: If X is a space and F is a closed subspace of X, then $c(F) \leq c(X)$. (c) For each of the invariants c you defined in (a), prove or disprove this statement: If X is a metrizable space and F is a closed subspace of X, then $c(F) \leq c(X)$.

(7) Prove or disprove these two statements. (a) The product of two Lindelof spaces is a Lindelof space. (b) The product of two metrizable Lindelof spaces is a Lindelof space.

(8) To begin, recall these two definitions. (a) A *generalized Cantor space* is a space (homeomorphic to) $\{0,1\}^\kappa$; here, $\{0,1\}$ is the 2-point discrete space, and κ is a cardinal number. (b) A space is *zero-dimensional* if it has a base of open-and-closed subsets. Now, outline a proof that a space is zero-dimensional if and only if it is homeomorphic to a subspace of a generalized Cantor space.

(9) For each ordinal ξ, let $W(\xi)$ denote the set of ordinals less than ξ in its usual order topology. [Restated: $W(\xi) = \xi$ as sets, and the topology of $W(\xi)$ is the topology for which $\{A(\eta) : \eta < \xi\} \cup \{B(\eta) : \eta < \xi\}$ is a subbase. Here for $\eta < \xi$, $A(\eta) := \{\sigma : \eta < \sigma < \xi\}$ and $B(\eta) := \{\sigma : \sigma < \eta\}$]. (a) For which ordinals ξ is $W(\xi)$ compact? Prove your assertion. (b) For which ordinals ξ is $W(\xi)$ countably compact? Prove your assertion.

(10) Many familiar principles are equivalent to AC (The Axiom of Choice). Among the equivalents are these five: Zorn's Lemma; Tukey's Lemma; the well-ordering principle; the Hausdorff maximality principle; König's Theorem. (a) State AC and any two of those five principles. (b) Choose any one of those five principles—let us call it P—and prove either that $AC \Rightarrow P$ or that $P \Rightarrow AC$.

TOPOLOGY PRELIMINARY EXAM

June 30, 2003

Directions. Do as much as you can; you are not expected to finish. The questions are weighted equally, though they may differ in difficulty. Do not spend too much time on one problem. You must be the judge of how much detail to provide.

1. Prove: For any set X, there is *no* surjection from X onto $\mathcal{P}(X)$ = the set of all subsets of X.

2. The fundamental group $\pi_1(X,x)$ is the set of pointed homotopy classses of pointed maps from (S^1,p) to (X,x). What is the set of pointed homotopy classes of pointed maps from (S^0,p) to (X,x), where S^0 is the two-point set $\{-1,1\}$ and $p = 1$?

3. Characterize each of the following, in terms of familiar properties. (Provide some justification for your answers.)

   a) Subspaces of Hausdorff spaces.
   b) Subspaces of compact spaces. ("compact" means only "each open cover has a finite subcover".)
   c) Subspaces of compact Hausdorff spaces.
   d) Open subspaces of compact Hausdorff spaces.
   e) Subspaces of path-connected spaces.

4. (a) List all the compact connected surfaces, up to homeomorphism.
   (b) How do you know that no two of the surfaces you have listed are homeomorphic? (Some details, please)
   (c) The connected sum of a torus and a projective plane is a compact connected surface. To which element of your list is it homeomorphic, and how do you know?

5. Show that the number of points in a connected, normal, T_1 space X is either ≤ 1 or $\geq 2^{\aleph_0}$.

6. Describe the universal cover of the Klein bottle.

7. Let X be locally compact and Hausdorff.

   a) Describe the one-point compactification X^* of X.

   b) Suppose also that X has a countable base. Show that X^* is metrizable.

8. Suppose that X is simply connected. Provide a proof of, or counterexample to, the following statement: Every map from X to the circle S^1 is homotopic to a constant map.

9. The Sorgenfry line S is the set of real numbers with $\{[a,b) \mid a < b\}$ as base for the topology.

a) Explain why S is completely regular, separable, Lindelöf, not metrizable.
b) Explain why $S \times S$ is separable, not normal, not Lindelöf.

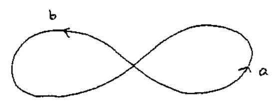
10. Prove: If $f : X \to Y$ is continuous and onto, with X compact Hausdorff and Y Hausdorff, then there is a subspace F of X minimal among compact subspaces Z of X with $f(Z) = Y$.

Wesleyan University
Doctoral Qualifying Topology Exam in Mathematics
Profs. Adeboye and Collins, August 7, 2013

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts. Please choose **at least two problems from each part and at least five questions total.** Always explain your reasoning. Good luck!

Part II

1. Let a and b be the generators of $\pi(S^1 \vee S^1)$. Draw a picture of a covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2 and ab.

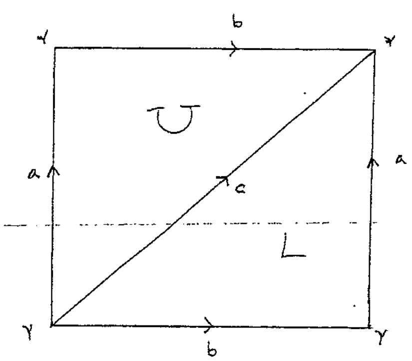


2. Consider a cylinder $S^1 \times I$. Identify the antipodal points of the boundary circle $S^2 \times \{0\}$. Also identify the antipodal points of the boundary circle $S^1 \times \{1\}$. Calculate the fundamental group of this surface.

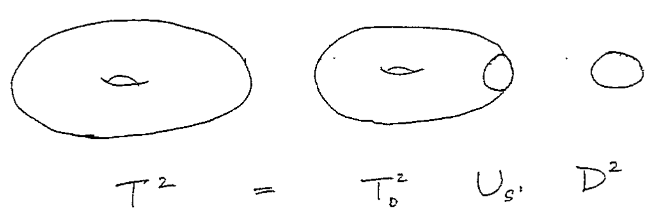
3. Consider the space obtained from two tori $T^2 = S^1 \times S^1$ by identifying the circle $S^1 \times \{(0,1)\}$ in one with $S^1 \times \{(0,1)\}$ in the other. Calculate the fundamental group of this space.

4. (a) Calculate the simplicial homology groups of S^1.

   (b) Calculate the simplicial homology groups of the torus using the Δ complex given below.



   (c) Let T_0^2 denote the the torus with the interior of a disk removed. Use the Mayer-Vietoris sequence for the torus, viewed as T_0^2 attached to a disk D^2 along a circle, to calculate the homology groups of T_0^2.



Qualifying exam in Topology, June 2006

You have 3 hours. Do as many as you can. No one is expected to create a perfect score. You are to be the judge of how much detail to provide in responding to "why?", "explain", "prove", "show", "sketch", etc. Try to allocate time roughly as .5 to part A and .5 to part B.

Part A

1. Let X be a space. The density character δX is defined as $\delta X = min\{m \mid X \text{ has a dense set of cardinal } m\}$.

   (a) Why does that "min" exist?

   Let $C(X) = \{f \in \mathbf{R}^X \mid f \text{ is continuous}\}$ and let m be infinite.

   (b) Prove: $\delta X \leq m$ implies $|C(X)| \leq 2^m$.

2. Let $\{X_a \mid a \in A\}$ (or just $\{X_a\}_A$) be a set of spaces. (A is arbitrary, not necessarily finite.)

   (a) Define the topology of the topological (Tychonoff) product $\Pi\{X_a \mid a \in A\}$ (or just $\Pi_A X_a$).

   (b) Here is the definition of topological sum $\sum\{X_a \mid a \in A\}$ (or just $\sum_A X_a$); we assume $a_1 \neq a_2 \Rightarrow X_{a_1} \cap X_{a_2} = \emptyset$. $\sum_A X_a$ is the set $\cup_A X_a$; $G \subseteq \cup_A X_a$ is "open" exactly when $G \cap X_a$ is open in X_a, $\forall a \in A$. Verify (quickly) that this gives us a topology.

3. A topological property is a class P of spaces for which ($X \in P$, Y homeomorphic to $X \Rightarrow Y \in P$). Consider these 6 topological properties: finite, countable, connected, locally connected, metrizable, normal. For each of these topological properties, answer the questions below.

   (a) For exactly what $|A|$ is it always true that $\{X_a\}_A \subseteq P \Rightarrow \Pi_A X_a \in P$? Explain.

   (b) For exactly what $|A|$ is it always true that $\{X_a\}_A \subseteq P \Rightarrow \sum_A X_a \in P$? Explain.

4. Let C be a class of spaces. A space I is called injective in C if $I \in C$ and whenever $X, Y \in C$, $X \subseteq Y$ and $f : X \to I$ is continuous, then there exists continuous $\overline{f} : Y \to I$ with $\overline{f}|_X = f$. Let C = all compact Hausdorff spaces. Explain why $[0,1]$ is injective in C.

5. For X a space, $clop(X) = \{U \subseteq X \mid U \text{ is closed and open}\}$. X is called zero-dimensional, zd, if $clop(X)$ is a base. Recall that the weight of a space Y, wY, is the minimum cardinal of a base.

   (a) Let $\mathbf{Q}$ be the rationals (with the usual topology). Show that $\mathbf{Q}$ is zd, and $w\mathbf{Q} < |clop(\mathbf{Q})|$.

   (b) Prove: if X is compact Hausdorff zd, then $wX = |clop(X)|$.

6. Recall from algebra the definition of a commutative ring with identity, R, (or $(R, +, \cdot, 0, 1)$), and that $I \subseteq R$ is called an ideal if I is a subring and $R \cdot I \subseteq I$.

   (a) Use Zorn's Lemma to show that each proper ideal is contained in a maximal ideal ("maximal" with respect to the partial order "$\subseteq$ of ideals").

Note that $C(X)$ (see 1.) is a ring, with $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$ and 0 and 1 the constant functions. For $f \in C(X)$, $Z(f) \equiv \{x \mid f(x) = 0\}$.

   (b) Prove: for any X, $S \subseteq X$, $I(S) \equiv \{f \in C(X) \mid Z(f) \supseteq X\}$ is an ideal in $C(X)$. Note that $I(\overline{S}) = I(S)$, and $S_1 \subseteq S_2 \Rightarrow I(S_1) \supseteq I(S_2)$.

   (c) Prove: for compact X, the converse holds: if I is an ideal in $C(X)$, then $I = I(S)$ for some closed S.

   (d) What are the maximal ideals in $C(X)$ for compact X? Explain.

7. Prove: if X is locally compact, and X is dense in Hausdorff Y, then X is open in Y.

8. Recall the construction and characteristic properties of the Čech-Stone compactification βX, for X completely regular and Hausdorff. Recall also zd from 5. A simpler version of the argument constructing βX will prove: If X is zd Hausdorff, then there is compact zd Hausdorff zX containing X densely such that: if $f : X \to \{0,1\}$ (discrete) is continuous, then there is unique $\overline{f} : zX \to \{0,1\}$ continuous with $\overline{f}|_X = f$. Sketch the proof of this.

Part B

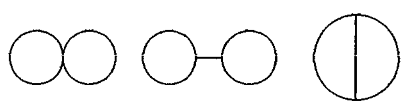
1. Let $f, f' : I \to X$.
   (a) Define the term "path homotopy" as applied to f, f'.
   (b) Show that the relation of path homotopy among maps from I to X is an equivalence relation.
   (c) Let $X = \mathbf{R}^2 - \{(0,0)\}$. Let $f : I \to X$ be

   $$f(t) = (\cos^2(2\pi t), \sin^2(2\pi t))$$

   Sketch the graph of f and determine for which integers k, f is path homotopic to $g_k : I \to X$ where

   $$g_k(t) = (\cos(2\pi kt), \sin(2\pi kt))$$

2. Let $g : X \to Y$ and $h : Y \to X$.
   (a) Define the term "homotopy equivalence" as applied to g and h.
   (b) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.
   (c) Prove that the three spaces below are homotopy equivalent.

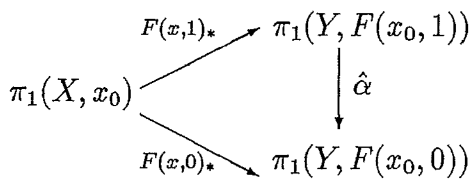


3. Let X be a topological space and $A \subset X$.
   (a) Define the term "deformation retract" as applied to A and X.
   (b) Construct an explicit deformation retraction of $\mathbf{R}^n - \{0\}$ onto S^{n-1}.

4. Let X be a topological space, and given $x \in X$, let c_x denote the constant path $c_x : I \to X$ carrying all of I to the point x. Let f be a path in X from x_0 to x_1. Prove, by **constructing a specific path homotopy** between them, both algebraically and with a diagram, that

$$[f] * [e_{x_1}] = [f] \text{ and } [e_{x_0}] * [f] = [f]$$

5. Let $F : X \times I \to Y$ be a continuous map. Let $x_0 \in X$ and let $\alpha : I \to Y$ by $\alpha(t) = F(x_0, t)$. Prove that the following diagram of maps commutes.



6. Compute the fundamental group, with proof, of the spaces X, Y below.

   (a) Let X be the space which is two copies of the cylinder $S^1 \times I$ identified at one point.

   (b) Let Y be the space B^2 with 2 points on its boundary identified.

7. (a) Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X, there exists a neighborhood $V \subset U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

   (b) Let X be the subspace of $\mathbf{R}^2$ consisting of the horizontal line segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

8. (a) State the Lebesgue number lemma.

   (b) Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Prove that any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

(c) Prove that the x-axis is a closed subspace of X whose subspace topology is the discrete topology.

(d) State the Tietze extension theorem, and use it to prove that X is not normal.

(10) Distinguishing surfaces

Let T_n be the n-fold torus (sometimes known as the torus with n holes), where $n \geq 1$. Prove that if p and q are distinct natural numbers, then T_p and T_q are not homeomorphic.

TOPOLOGY PRELIMINARY EXAM
June 30, 2003

Directions. Do as much as you can; you are not expected to finish. The questions are weighted equally, though they may differ in difficulty. Do not spend too much time on one problem. You must be the judge of how much detail to provide.

1. Prove: For any set X, there is *no* surjection from X onto $\mathcal{P}(X)$ = the set of all subsets of X.

2. The fundamental group $\pi_1(X,x)$ is the set of pointed homotopy classses of pointed maps from (S^1,p) to (X,x). What is the set of pointed homotopy classes of pointed maps from (S^0,p) to (X,x), where S^0 is the two-point set $\{-1,1\}$ and $p = 1$?

3. Characterize each of the following, in terms of familiar properties. (Provide some justification for your answers.)

   a) Subspaces of Hausdorff spaces.
   b) Subspaces of compact spaces. ("compact" means only "each open cover has a finite subcover".)
   c) Subspaces of compact Hausdorff spaces.
   d) Open subspaces of compact Hausdorff spaces.
   e) Subspaces of path-connected spaces.

4. (a) List all the compact connected surfaces, up to homeomorphism.
   (b) How do you know that no two of the surfaces you have listed are homeomorphic? (Some details, please)
   (c) The connected sum of a torus and a projective plane is a compact connected surface. To which element of your list is it homeomorphic, and how do you know?

5. Show that the number of points in a connected, normal, T_1 space X is either ≤ 1 or $\geq 2^{\aleph_0}$.

6. Describe the universal cover of the Klein bottle.

7. Let X be locally compact and Hausdorff.

   a) Describe the one-point compactification X^* of X.

   b) Suppose also that X has a countable base. Show that $X^*$is metrizable.

8. Suppose that X is simply connected. Provide a proof of, or counterexample to, the following statement: Every map from X to the circle S^1 is homotopic to a constant map.

9. The Sorgenfry line S is the set of real numbers with $\{[a, b) \mid a < b\}$ as base for the topology.

a) Explain why S is completely regular, separable, Lindelöf, not metrizable.
b) Explain why $S \times S$ is separable, not normal, not Lindelöf.

10. Prove: If $f : X \to Y$ is continuous and onto, with X compact Hausdorff and Y Hausdorff, then there is a subspace F of X minimal among compact subspaces Z of X with $f(Z) = Y$.