Topology Preliminary Exam

June 11, 2014

1. (a) Prove that the image of a connected space under a continuous map is connected.

   (b) Prove that the image of a compact space under a continuous map is compact.

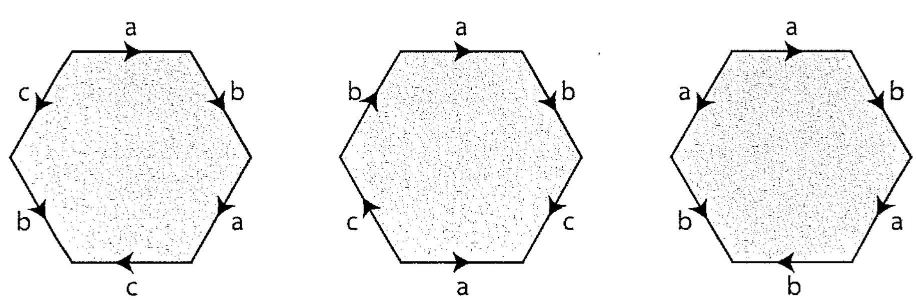
2. Show that the retract of a contractible space is contractible.

3. Let $f, g : X \to Y$ be continuous functions and assume that Y is Hausdorff. Show that $\{x | f(x) = g(x)\}$ is closed in X.

4. Let $X = D^2 \times S^1$ be a solid torus, and let $A = S^1 \times S^1$ be its boundary. Compute $H_n(X, A)$ for all n.

5. Let $C[0, 1]$ denote the set of all continuous functions $f : [0, 1] \to \mathbb{R}$.
   The functions
   $$\rho(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$
   and
   $$\mu(f, g) = \max_{x\in[0,1]} \{|f(x) - g(x)|\}$$
   define metrics on $C[0, 1]$. Which metric induces the finer topology?

6. Consider the following three identification spaces of the disk.



   (a) Compute the fundamental group of each of these spaces.

   (b) Compute the homology groups of each of these spaces.

   (c) Which of these are manifolds?

Wesleyan University

Doctoral Qualifying Topology Exam in Mathematics

Profs. Adeboye and Collins, August 7, 2013

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts. Please choose **at least two problems from each part and at least five questions total.** Always explain your reasoning. Good luck!

Part I

1. Let ω be the cardinality of the natural numbers, and R be the set of real numbers. Let

$$U = \{-2 \le x < 0\} \cup \{1 \le x \le 2\} \subseteq \mathrm{R}$$

with the subspace topology.

Which of the properties below does U^ω with the product topology have? Give a reason for each.

   (a) connected
   (b) path-connected
   (c) compact
   (d) Hausdorff
   (e) first countable
   (f) metrizable
   (g) completely regular
   (h) normal

2. (a) Let X be a set and $(X, T_1), (X, T_2)$ be two topologies on X. Suppose that T_1 is finer than T_2 (that is, every open set in T_2 is also in T_1). Let (X, T_1) be compact and (X, T_2) be Hausdorff. Prove that $T_1 = T_2$.

   (b) Let $I = [0, 1]$. Let (I, K) be the topology of I as a subspace of R_K, and let (I, L) be the topology of I as a subspace of R_ℓ. Let $X = \prod_{\alpha \in J} Y_\alpha$ where Y_α is either (I, K) or (I, L) for each α (if $\alpha \neq \beta$, then Y_α need not equal Y_β), with the product topology. Use (a) to prove that X is not compact.

3. Let C_r be the circle of radius r, centered at the origin, in R^2. Determine the connected components, path-connected components and quasi-connected components of the subspaces of R^2 below.

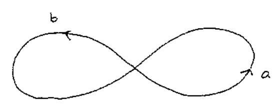
   (a) $S = \{C_{\frac{1}{n}} \mid n \in \mathrm{Z}_+\} \cup \{(0,0), (1,1)\}$
   (b) $T = \{C_r \mid r \in \mathrm{Q} \text{ and } 0 < r < 1\} \cup \{(-1,0), (0,0), (1,0)\}$

4. (a) State the Urysohn Metrization Theorem. Illustrate the theorem with two examples.

   (b) Describe in detail a metric space which does not satisfy the hypotheses of the Urysohn Metrization Theorem.

Part II

1. Let a and b be the generators of $\pi(S^1 \vee S^1)$. Draw a picture of a covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2 and ab.

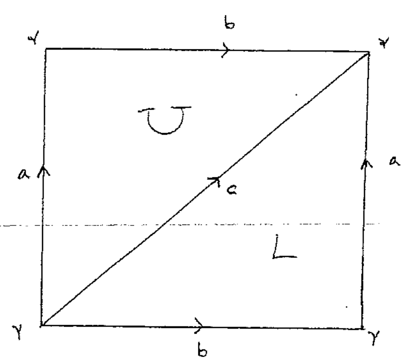


2. Consider a cylinder $S^1 \times I$. Identify the antipodal points of the boundary circle $S^2 \times \{0\}$. Also identify the antipodal points of the boundary circle $S^1 \times \{1\}$. Calculate the fundamental group of this surface.

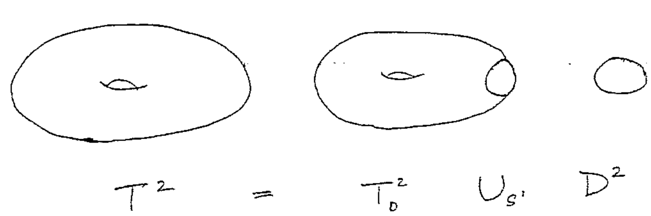
3. Consider the space obtained from **two** tori $T^2 = S^1 \times S^1$ by identifying the circle $S^1 \times \{(0,1)\}$ in one with $S^1 \times \{(0,1)\}$ in the other. Calculate the fundamental group of this space.

4. (a) Calculate the simplicial homology groups of S^1.

   (b) Calculate the simplicial homology groups of the torus using the Δ complex given below.



   (c) Let T_0^2 denote the the torus with the interior of a disk removed. Use the Mayer-Vietoris sequence for the torus, viewed as T_0^2 attached to a disk D^2 along a circle, to calculate the homology groups of T_0^2.



Wesleyan University

Department of Mathematics

PhD Qualifying Exam, Written Part: Topology

Profs. Leidy and Rasmussen

August 15, 2012

This exam consists of eight questions. You have three hours to complete this exam. You may not use any outside sources, and you should work alone.

You should do as much as you can of the exam in the allotted time, which does not mean that we expect you to finish every problem. Partial credit will be awarded when significant progress on a problem has been achieved.

1. Several statements are given below regarding either a pair of path-connected topological spaces X and Y, or a path-connected topological space X. (We consider only path-connected spaces so that we do not have to worry about basepoints.) For each statement, either give an explicit example of path-connected space(s) that satisfy the statement, or state that no example exists.

   a. X is homotopy equivalent to Y, but X is not homeomorphic to Y.

   b. X is homotopy equivalent to Y, but X is not a deformation retract of Y and Y is not a deformation retract of X.

   c. X is a deformation retract of Y, but X is not homotopy equivalent to Y.

   d. X is a retract of Y, but X is not a deformation retract of Y.

   e. X is a deformation retract of Y, but X is not a retract of Y.

   f. X is homotopy equivalent to Y, but $\pi_1(X) \ncong \pi_1(Y)$.

   g. $\pi_1(X) \cong \pi_1(Y)$, but X is not homotopy equivalent to Y.

   h. X is a deformation retract of Y, but $\pi_1(X) \ncong \pi_1(Y)$.

   i. X is contractible, but $\pi_1(X)$ is not trivial.

   j. $\pi_1(X)$ is trivial, but X is not contractible.

   k. $H_1(X) \cong H_1(Y)$, but $\pi_1(X) \ncong \pi_1(Y)$.

   l. $\pi_1(X) \cong \pi_1(Y)$, but $H_1(X) \ncong H_1(Y)$.

   m. $H_2(X) \cong H_2(Y)$, but $\pi_1(X) \ncong \pi_1(Y)$.

   n. $\pi_1(X) \cong \pi_1(Y)$, but $H_2(X) \ncong H_2(Y)$.

2. Suppose that

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

   is a short-exact sequence of chain complexes of abelian groups. Describe the induced long-exact sequence on homology, including a definition for all maps in the long-exact sequence. If necessary, show your definitions are well-defined. (However, you do not need to show the sequence is exact.)

3. Prove or disprove each of the following statements.

   a. The image of a Hausdorff space under a continuous map is Hausdorff.

   b. The image of a connected space under a continuous map is connected.

4. a. Define the notion of a CW-**complex**.

   b. Give an example of a CW-complex for the spaces $\mathbb{RP}^2$ and $\mathbb{CP}^4$.

   c. Compute the homology of $\mathbb{CP}^4$, using the CW-complex structure you gave in part b.

5. a. Show that every closed subspace of a compact space is compact.

   b. Show that every compact subspace of a Hausdorff space is closed.

6. Let A be a discrete set of $r > 1$ points in S^2. Determine the relative homology groups $H_n(S^2, A)$ for all $n \geq 0$.

7. Prove or disprove each of the following statements.

   a. If A is a subspace of X and X is simply connected, then A is simply connected.

   b. If $r : X \to A$ is a retraction, then $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.

8. Let X be the Klein bottle.

   a. Give a Δ-complex structure on X.

   b. Using the structure in part a, compute the simplicial cohomology groups of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Wesleyan University

Department of Mathematics
PhD Qualifying Exam, Written Part: Topology
Profs. Hovey and Scowcroft
August 10, 2011

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts. Please choose **at least two problems from each part and at least six questions total.** Good luck!

Part I

(1) State definitions of the following:
  (a) product topology
  (b) normal space
  (c) regular space
  (d) second-countable space.
  And state the following theorems:
  (i) Tychonoff Product Theorem
  (ii) Urysohn's Lemma
  (iii) Urysohn's Metrization Theorem
  (iv) Tietze's Extension Theorem.

(2) Show that every connected metric space with more than one point is uncountable.

(3) (a) State Zorn's Lemma.
  (b) Use Zorn's Lemma to prove that every commutative ring with identity $1 \neq 0$ has a maximal ideal.

(4) When X is a topological space, $C(X)$ is the ring of all continuous real-valued functions on X (with the usual operations: for example, $(f+g)(y) = f(y) + g(y)$ for all $f, g \in C(X)$ and $y \in X$). When $f \in C(X)$ and $S \subseteq X$, let

$$Z(f) = \{p \in X : f(p) = 0\}$$

and

$$I(S) = \{f \in C(X) : S \subseteq Z(f)\}.$$

  (a) When $S \subseteq X$, show that $I(S)$ is an ideal of $C(X)$ and that $I(S) = I(\mathrm{cl}(S))$. When $S_1 \subseteq S_2 \subseteq X$, show that $I(S_2) \subseteq I(S_1)$.
  (b) If X is completely regular and $p \in X$, show that $I(\{p\})$ is a maximal ideal of $C(X)$.
  (c) If X is a compact Hausdorff space and I is a maximal ideal of $C(X)$, show that $I = I(\{p\})$ for some $p \in X$.

(over)

Part II

(5) (a) Give a detailed definition of the singular homology group $H_p(X)$ of a topological space X, where p is a fixed positive integer.
(b) Find a CW complex X with $H_5(X) = \mathbb{Z}/2011\mathbb{Z}$, $H_6(X) = \mathbb{Z}$, and all other positive-dimensional homology groups being 0.
(c) State the Mayer-Vietoris theorem.

(6) (a) Prove the simplest case of the van Kampen theorem: namely, that if $X = U \cup V$ where U and V are open and simply connected, and $U \cap V$ is nonempty and path connected, then X is simply connected.
(b) State (precisely) the full van Kampen theorem.

(7) Suppose X is a Hausdorff space that has a basis of sets that are both open and closed. Show that X is totally disconnected: that is, that the only connected subsets of X are one-point sets.

(8) Let X be the 2-holed torus with a little disc removed. Let A be the boundary of the little disc that was removed. Calculate the maps induced by the inclusion $A \to X$ on the fundamental group and on homology. Use this to prove that A is not a retract of X.

(9) Suppose X is the space S^2 together with the straight-line segment from the north pole to the south pole. Find a simply-connected covering space of X and use this to compute the fundamental group of X.

Wesleyan University

Department of Mathematics
PhD Qualifying Exam, Written Part: Topology
Profs. Collins and Leidy
August 11, 2010

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts. Please choose at least two problems from each part and at least five questions total. Good luck!

Part I

1. Let $I = [0,1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

2. (a) Which of the properties below does $\mathbb{R}^n$ have? Give a reason for each.
   i. connected
   ii. path-connected
   iii. compact
   iv. Hausdorff
   v. completely regular
   vi. normal
   vii. metrizable

   (b) Which of the properties below does $\overline{S_\Omega}$ have? Give a reason for each.
   i. connected
   ii. path-connected
   iii. compact
   iv. Hausdorff
   v. completely regular
   vi. normal
   vii. metrizable

3. (a) State the Urysohn metrization theorem.

   (b) Prove that the continuous image of a compact metric space in a Hausdorff space is metrizable.

4. (a) Give an example of space which has infinitely many elements, is connected but not path-connected, nor metrizable.

   (b) Give an example of space which has infinitely many elements, is normal, Hausdorff and first countable, but not metrizable, nor connected.

   (c) Give an example of space which has infinitely many elements and is metrizable, first countable and second countable.

   (d) Give an example of space which has infinitely many elements, is completely regular, but not compact, connected or first countable.

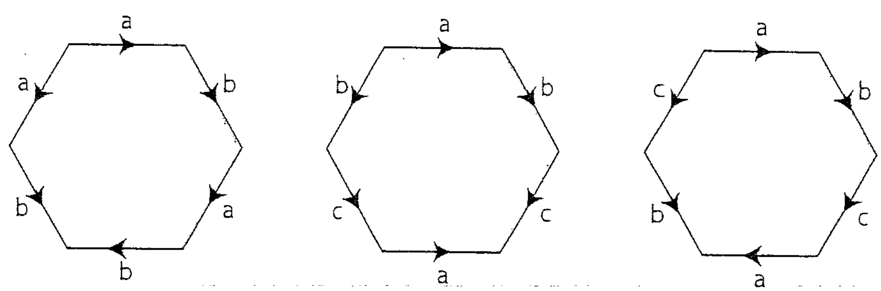
Part II

1. (a) Describe a space X such that $\pi_1(X) \cong \langle x, y, z | xyx^{-1}y^{-1}, z^2 \rangle$.
   (b) Describe a space X such that $\pi_1(X) \cong \langle x, y | x^2, y^2, xyx^{-1}y^{-1} \rangle$.
   (c) Describe two (connected) spaces X and Y such that $H_i(X) \cong H_i(Y)$, for all i, but $\pi_1(X) \ncong \pi_1(Y)$.

2. Let $(\widetilde{X}, p)$ be a covering space of X.
   (a) Show that if X is Hausdorff, then $\widetilde{X}$ is Hausdorff.
   (b) Show that if X is compact and $p^{-1}(x)$ is finite for each $x \in X$, then X is compact.

3. (a) A (thickened) knot is an embedding $K : S^1 \times D^2 \to S^3$. The knot complement is $S^3 - \text{im}(K)$. Calculate $H_n(S^3 - \text{im}(K))$ for any knot K.
   (b) A knot $K : S^1 \times D^2 \to S^3$ is slice if it extends to an embedding $\Delta : D^2 \times D^2 \to B^4$ where $S^1 = \partial D^2$ and $S^3 = \partial B^4$. The slice disk complement is $B^4 - \text{im}(\Delta)$. Calculate $H_n(B^4 - \text{im}(\Delta))$.

4. Consider the following three identification spaces of the disk.



   (a) Compute the fundamental group of each of these spaces.
   (b) Compute the homology groups of each of these spaces.
   (c) Which of these are manifolds?
   (d) Recall that every surface (compact, connected 2-manifold) is homeomorphic to S^2 or a connected sum of tori or a connected sum of real projective planes. For each space above that is a surface, determine which of these it is homeomorphic to.

Wesleyan University

Department of Mathematics

PhD Qualifying Exam, Written Part: Topology

Profs. Hovey and Leidy

August 6, 2009

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has two parts. Please choose **at least two problems from each part and at least six questions total.** Good luck!

Part I

(1) One's first exposure to topology is often through metric spaces (or, really, metrizable spaces). This problem asks why we can't just stick with metrizable spaces.

(a) Define a metric d and a metrizable space X.

(b) Give an example of a space that is not metrizable (with some justification).

(c) There are 3 basic constructions that build new spaces out of old ones in topology: (possibly infinite) products, subspaces, and quotient spaces. Under which of these operations are metrizable subspaces closed? (That is, are products of metrizable spaces metrizable? Subspaces? Quotient spaces? I want short explanations if the answer is yes and counterexamples if the answer is no).

(2) One of the common "weird" spaces we did not look at is the **slotted plane** X. This is $\mathbb{R}^2$ with an unusual topology. We define a set U to be open in the slotted plane if for every point $x \in U$, there is an open disk D around x of positive radius, a number $k \geq 0$, and straight lines $L_1, \ldots, L_k$ through x so that

$$\{x\} \cup (D - L_1 - L_2 - \cdots - L_k) \subseteq U.$$

(a) Prove that the slotted plane is a topological space, with more open sets than the usual topology.

(b) Prove that the slotted plane is Hausdorff but not regular.

(c) Prove that the slotted plane is separable but not even first countable.

(3) All kinds of topologists agree that the unit interval $I = [0, 1]$ is crucially important. In general topology, we often consider all the spaces X we can build from I by taking subspaces of (possibly infinite) products of copies of I. Exactly what spaces X do we get by doing this? Some explanation of why your answer is true is needed, though a complete proof is not.

(4) If I is so important, we should look at $C(X, I)$, the set of continuous functions from X to I. Let's try to count them. Let c denote the cardinality of I.

(a) Prove that if X is nonempty, there are at least c functions in $C(X, I)$.

(b) Prove the collection of all functions from I to itself has more than c functions in it.

(c) Prove that $C(I, I)$ has exactly c functions in it. (Hint: continuity must mean that there is some smaller set A in I so if we know f on A then we know it on all of I). This means that the probability of a function from I to I being continuous is 0.

(5) One of the cool theorems about I that we did not get to is the Hahn-Mazurkiewicz theorem: If X is Hausdorff, then there is a continuous onto map $f\colon I \to X$ if and only if X is compact, connected, locally connected, and metrizable. In this problem, we will prove that X is at least metrizable. So suppose X is Hausdorff and there is a continuous onto map $f\colon I \to X$.

(a) Show that f is a closed map.

(b) Show that X is second countable. (Hint: choose a countable basis B for I. One's first thought might be to take $f(U)$ for U in B, but there is no guarantee these are open, so that won't work. But $f(I-U)$ is closed, so we can take $X - f(I-U)$ for U in B. Unfortunately, that is not quite enough. Let C be the set of all finite unions of elements of B, and take all of the $X - f(I-U)$ for U in C. Prove this is a basis for X).

(c) Prove that X is metrizable.

Part II

(1) Prove or disprove each of the following statements.

(a) If A is a subspace of X and X is simply connected, then A is simply connected.

(b) If $r : X \to A$ is a retraction, then $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.

(c) If A is a deformation retract of X and $i : A \to X$ is the inclusion map, then $i_* : \pi_1(A) \to \pi_1(X)$ is an isomorphism.

(2) Compute the fundamental group of each of the following.

(a) $M \times \mathbb{R}P(\infty)$, where M is the Möbius band.

(b) $\Sigma_2 \times \mathbb{C}P(2)$, where Σ_2 is the orientable surface of genus 2.

(c) $L(p,q) \vee K$, where K is the Klein bottle.

(d) The 1-skeleton of Δ^3, the standard 3-simplex.

(3) (a) Classify all of the 2-sheeted covering spaces of $S^1 \vee S^1$ up to covering equivalence. For each one, calculate the associated subgroup of $\pi_1(S^1 \vee S^1)$ and the deck group.

(b) Construct an example of a covering space of $S^1 \vee S^1$ that is not regular.

(c) What is the universal cover of $S^1 \vee S^1$?

(d) Recall that the universal abelian cover is the covering space corresponding to the commutator subgroup. What is the universal abelian cover of $S^1 \vee S^1$?

(4) (a) Let $X = D^2 \times S^1$ be a solid torus, and let $A = S^1 \times S^1$ be its boundary. Compute $H_n(X, A)$ for all n.

(b) A knot K is an embedding of S^1 into S^3. A knot is tame if there is an embedding $f : S^1 \times D^2 \to S^3$ such that f restricted to $S^1 \times \vec{0}$ is the knot. Denote $f(S^1 \times D^2)$ by $N(K)$. Compute $H_n(S^3 - N(K))$ for all n.

(5) Suppose the diagram below is commutative and that the rows are exact. Show that if α is surjective and β and δ are injective, then γ is injective.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} & & \\
0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' & \longrightarrow & 0
\end{array}$$

Topology Qualifying Exam
August 4, 2008

Instructions: The exam has two parts. Please read the instructions for each part. Briefly justify all answers for which no proof is specifically requested. The exam is closed book and will run from 9 a.m. to noon.

Part I: Complete each of the three problems.

1. Let X, Y be topological spaces. If $f : X \to Y$ is a continuous map and X is compact, show that $f(X)$ is compact.

2. Let X, Y be topological spaces. If X and Y are Hausdorff, show that $X \times Y$ is Hausdorff.

3. Let X be a topological space, and $f : I \to X$ be a path in X. Let $\overline{f}$ be the path defined by $\overline{f}(s) = f(1-s)$. Let e_{x_0} denote the constant path that carries all of I to the point $f(0)$.

   Construct a specific path homotopy to show that

$$[f] * [\overline{f}] = [e_{x_0}]$$

   The path homotopy must be described algebraically and in full detail, with proofs.

Part II: Complete at least two of the four problems.
Additional credit will be given for additional problems worked.

1. For each part, either give an example of connected (so we don't have to worry about basepoints) topological spaces X and Y that satisfy the statement, or state that no example exists.

   (a) X is homotopy equivalent to Y, but X is not homeomorphic to Y.

   (b) X is homotopy equivalent to Y, but X is not a deformation retract of Y and Y is not a deformation retract of X.

   (c) X is a deformation retract of Y, but X is not homotopy equivalent to Y.

   (d) X is homotopy equivalent to Y, but $\pi_1(X) \not\cong \pi_1(Y)$.

   (e) $\pi_1(X) \cong \pi_1(Y)$, but X is not homotopy equivalent to Y.

   (f) X is a retract of Y, but X is not a deformation retract of Y.

   (g) X is a deformation retract of Y, but X is not a retract of Y.

   (h) X is a deformation retract of Y, but $\pi_1(X) \not\cong \pi_1(Y)$.

   (i) $H_1(X) \cong H_1(Y)$, but $\pi_1(X) \not\cong \pi_1(Y)$

   (j) $\pi_1(X) \cong \pi_1(Y)$, but $H_1(X) \not\cong H_1(Y)$

   (k) $H_2(X) \cong H_2(Y)$, but $\pi_1(X) \not\cong \pi_1(Y)$

   (l) $\pi_1(X) \cong \pi_1(Y)$, but $H_2(X) \not\cong H_2(Y)$

2. (a) Determine the fundamental group of each space:
    i. the circle
    ii. the torus
    iii. the infinite cylinder $S^1 \times \mathbf{R}$
    iv. the subset of $\mathbf{R}^2$ which is $S^1 \cup (\mathbf{R} \times 0)$
    v. $\mathbf{R}^2 - (\mathbf{R} \times 0)$

   (b) Find the universal covering spaces of each space:
    i. the circle
    ii. the sphere
    iii. the projective plane
    iv. the torus

   (c) Compute the simplicial homology groups of each space:
    i. the circle
    ii. the sphere
    iii. the projective plane

3. State the definition of a path-connected space.

   Let X be a path-connected topological space and x_0, x_1 be two points of X. Describe in detail an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

4. State the definition of a covering space $p : \widetilde{X} \to X$.

   Let X and Y be topological spaces. Prove the following: Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$ with two lifts, $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ that agree at one point of Y, then if Y is connected, these two lifts must agree on all of Y.

Wesleyan University
Department of Mathematics

PhD Qualifying Exam, Written Part: Topology
Profs. Davis and Hovey

June 13, 2007

This is a 3-hour exam; no books or notes or consultations are permitted. We do not expect you to do all of the problems. Good luck!

(1) $\mathbb{R}^n$.
   (a) Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$.
   (b) Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 2$.

(2) Computing the fundamental group
   In this problem, a "triangle" consists of only three edges and three vertices and does not include the interior region (inside the edges and vertices). Let T_1 be the triangle in the plane with vertices $(5,5)$, $(7,5)$, $(6,6)$; let T_2 be the triangle with vertices $(-5,-5)$, $(-6,-6)$, $(-4,-6)$; and let T_3 be the triangle with vertices $(17,-5)$, $(18,-6)$, $(16,-6)$. Also, let P be the line segment from $(-5,-5)$ to $(5,5)$ that includes the endpoints, and let Q be the line segment from $(7,5)$ to $(17,-5)$ that includes the endpoints. Let X be the subspace of $\mathbb{R}^2$ that is the union of T_1, T_2, T_3, P, and Q. Find, with proof, $\pi_1(X,(0,0))$.

(3) Products
   (a) Suppose X_i are topological spaces for $i \in I$. Define the product and box topologies on $\prod_i X_i$. Define the uniform topology on $\prod_i X_i$ if the X_i are assumed to be metric spaces.
   (b) There is one property that is the reason the product topology is the most important of all topologies on $\prod_i X_i$. What is it?
   (c) Give an example (with some proof) of a sequence that converges in the product topology but not in the uniform or box topologies.

(4) Surfaces
   For this problem, let X be the quotient space of six disjoint polygonal regions with labelling scheme

$$bc^{-1}a,\ abc,\ ded^{-1}ef,\ fgh,\ g^{-1}hik,\ ji^{-1}jk^{-1}.$$

   (a) List all the compact connected surfaces up to homeomorphism.
   (b) Identify, with justification, X in terms of the surfaces in (a).

(5) Subspaces.
   (a) Give an internal condition on X so that X satisfies the condition if and only if X is an open subspace of a compact Hausdorff

space. Some explanation is required, though not a complete proof.

(b) Give an internal condition on X so that X satisfies the condition if and only if X is an arbitrary subspace of a compact Hausdorff space X. Similarly, some explanation is required.

(c) Given an internal condition on X so that X satisfies the condition if and only if X is an arbitrary subspace of a simply connected space. (Simply connected means path connected and trivial fundamental group). Again, some explanation is required.

(6) Homotopy equivalences

Let $f: X \to Y$ be a homotopy equivalence.

(a) Prove that f induces a bijection between the set of path components of X and the set of path components of Y.

(b) Also, prove that f restricts to a homotopy equivalence from each path component of X to the corresponding path component of Y (under the bijection in part (a)).

(7) Separability.

(a) Prove that every second-countable topological space is separable (i.e. has a countable dense subset).

(b) Prove that $\mathbb{R}_\ell$ ($\mathbb{R}$ with the lower-limit topology; the Sorgenfrey line) is separable but not second countable.

(c) Prove that, if X is separable, then there are no more than c continuous functions from X to $\mathbb{R}$, where c denotes the cardinality of $\mathbb{R}$.

(8) Covering spaces

Explain the covering space proof of the fundamental group of the circle S^1, writing the proof so it will apply to as general a situation as possible (but, of course, no more general than is possible).

(9) A new example: the Moore plane

(a) Define a topology on the upper half-plane

$$X = \{(p,q)|q \geq 0\} \subseteq \mathbb{R}^2$$

by defining a basis to be the open disks $B((p,q);\epsilon)$ centered at (p,q) with radius ϵ for $0 < \epsilon < q$ (so the disk stays above the x-axis) and the sets $B((p,q),q) \cup \{(p,0)\}$ (so the disk is tangent to the x-axis at $(p,0)$ and contains the point $(p,0)$ but no other point of its boundary). Prove that this is a basis for a topology.

(b) Prove that the set

$$C = \{(p,q)|q > 0 \text{ and } p,q \text{ rational}\}$$

is dense in X, so that X is separable. According to part (c) of the separability problem, then, X has at most c continuous functions to $\mathbb{R}$.

(c) Prove that the x-axis is a closed subspace of X whose subspace topology is the discrete topology.
(d) State the Tietze extension theorem, and use it to prove that X is not normal.

(10) Distinguishing surfaces

Let T_n be the n-fold torus (sometimes known as the torus with n holes), where $n \geq 1$. Prove that if p and q are distinct natural numbers, then T_p and T_q are not homeomorphic.

Qualifying exam in Topology, June 2006

You have 3 hours. Do as many as you can. No one is expected to create a perfect score. You are to be the judge of how much detail to provide in responding to "why?", "explain", "prove", "show", "sketch", etc. Try to allocate time roughly as .5 to part A and .5 to part B.

Part A

1. Let X be a space. The density character δX is defined as $\delta X = min\{m \mid X$ has a dense set of cardinal $m\}$.

   (a) Why does that "min" exist?

   Let $C(X) = \{f \in \mathbf{R}^X \mid f$ is continuous$\}$ and let m be infinite.

   (b) Prove: $\delta X \leq m$ implies $|C(X)| \leq 2^m$.

2. Let $\{X_a \mid a \in A\}$ (or just $\{X_a\}_A$) be a set of spaces. (A is arbitrary, not necessarily finite.)

   (a) Define the topology of the topological (Tychonoff) product $\Pi\{X_a \mid a \in A\}$ (or just $\Pi_A X_a$).

   (b) Here is the definition of topological sum $\sum\{X_a \mid a \in A\}$ (or just $\sum_A X_a$); we assume $a_1 \neq a_2 \Rightarrow X_{a_1} \cap X_{a_2} = \emptyset$. $\sum_A X_a$ is the set $\cup_A X_a$; $G \subseteq \cup_A X_a$ is "open" exactly when $G \cap X_a$ is open in X_a, $\forall a \in A$. Verify (quickly) that this gives us a topology.

3. A topological property is a class P of spaces for which ($X \in P$, Y homeomorphic to $X \Rightarrow Y \in P$). Consider these 6 topological properties: finite, countable, connected, locally connected, metrizable, normal. For each of these topological properties, answer the questions below.

   (a) For exactly what $|A|$ is it always true that $\{X_a\}_A \subseteq P \Rightarrow \Pi_A X_a \in P$? Explain.

   (b) For exactly what $|A|$ is it always true that $\{X_a\}_A \subseteq P \Rightarrow \sum_A X_a \in P$? Explain.

4. Let C be a class of spaces. A space I is called injective in C if $I \in C$ and whenever $X, Y \in C$, $X \subseteq Y$ and $f : X \to I$ is continuous, then there exists continuous $\overline{f} : Y \to I$ with $\overline{f}|_X = f$. Let $C =$ all compact Hausdorff spaces. Explain why $[0,1]$ is injective in C.

5. For X a space, $clop(X) = \{U \subseteq X \mid U \text{ is closed and open}\}$. X is called zero-dimensional, zd, if $clop(X)$ is a base. Recall that the weight of a space Y, wY, is the minimum cardinal of a base.

   (a) Let $\mathbf{Q}$ be the rationals (with the usual topology). Show that $\mathbf{Q}$ is zd, and $w\mathbf{Q} < |clop(\mathbf{Q})|$.

   (b) Prove: if X is compact Hausdorff zd, then $wX = |clop(X)|$.

6. Recall from algebra the definition of a commutative ring with identity, R, (or $(R, +, \cdot, 0, 1)$), and that $I \subseteq R$ is called an ideal if I is a subring and $R \cdot I \subseteq I$.

   (a) Use Zorn's Lemma to show that each proper ideal is contained in a maximal ideal ("maximal" with respect to the partial order "$\subseteq$ of ideals").

   Note that $C(X)$ (see 1.) is a ring, with $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$ and 0 and 1 the constant functions. For $f \in C(X)$, $Z(f) \equiv \{x \mid f(x) = 0\}$.

   (b) Prove: for any X, $S \subseteq X$, $I(S) \equiv \{f \in C(X) \mid Z(f) \supseteq X\}$ is an ideal in $C(X)$. Note that $I(\overline{S}) = I(S)$, and $S_1 \subseteq S_2 \Rightarrow I(S_1) \supseteq I(S_2)$.

   (c) Prove: for compact X, the converse holds: if I is an ideal in $C(X)$, then $I = I(S)$ for some closed S.

   (d) What are the maximal ideals in $C(X)$ for compact X? Explain.

7. Prove: if X is locally compact, and X is dense in Hausdorff Y, then X is open in Y.

8. Recall the construction and characteristic properties of the Čech-Stone compactification βX, for X completely regular and Hausdorff. Recall also zd from 5. A simpler version of the argument constructing βX will prove: If X is zd Hausdorff, then there is compact zd Hausdorff zX containing X densely such that: if $f : X \to \{0, 1\}$ (discrete) is continuous, then there is unique $\overline{f} : zX \to \{0, 1\}$ continuous with $\overline{f}|_X = f$. Sketch the proof of this.

Part B

1. Let $f, f' : I \to X$.

   (a) Define the term "path homotopy" as applied to f, f'.

   (b) Show that the relation of path homotopy among maps from I to X is an equivalence relation.

   (c) Let $X = \mathbf{R}^2 - \{(0,0)\}$. Let $f : I \to X$ be

   $$f(t) = (\cos^2(2\pi t), \sin^2(2\pi t))$$

   Sketch the graph of f and determine for which integers k, f is path homotopic to $g_k : I \to X$ where

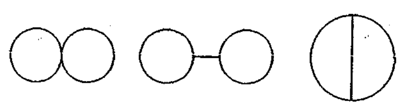
   $$g_k(t) = (\cos(2\pi kt), \sin(2\pi kt))$$

2. Let $g : X \to Y$ and $h : Y \to X$.

   (a) Define the term "homotopy equivalence" as applied to g and h.

   (b) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

   (c) Prove that the three spaces below are homotopy equivalent.



3. Let X be a topological space and $A \subset X$.

   (a) Define the term "deformation retract" as applied to A and X.

   (b) Construct an explicit deformation retraction of $\mathbf{R}^n - \{0\}$ onto S^{n-1}.

4. Let X be a topological space, and given $x \in X$, let c_x denote the constant path $c_x : I \to X$ carrying all of I to the point x. Let f be a path in X from x_0 to x_1. Prove, by **constructing a specific path homotopy** between them, both algebraically and with a diagram, that

   $$[f] * [e_{x_1}] = [f] \text{ and } [e_{x_0}] * [f] = [f]$$

5. Let $F : X \times I \to Y$ be a continuous map. Let $x_0 \in X$ and let $\alpha : I \to Y$ by $\alpha(t) = F(x_0, t)$. Prove that the following diagram of maps commutes.

$$
\begin{array}{ccc}
 & \overset{F(x,1)_*}{\nearrow} & \pi_1(Y, F(x_0,1)) \\
\pi_1(X, x_0) & & \big\downarrow \hat{\alpha} \\
 & \underset{F(x,0)_*}{\searrow} & \pi_1(Y, F(x_0,0))
\end{array}
$$

6. Compute the fundamental group, with proof, of the spaces X, Y below.

   (a) Let X be the space which is two copies of the cylinder $S^1 \times I$ identified at one point.

   (b) Let Y be the space B^2 with 2 points on its boundary identified.

7. (a) Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X, there exists a neighborhood $V \subset U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

   (b) Let X be the subspace of $\mathbf{R}^2$ consisting of the horizontal line segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

8. (a) State the Lebesgue number lemma.

   (b) Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Prove that any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.